PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Stability of Forced Steady Solitary Waves

R. Camassa and T. Yao-Tsu Wu

Phil. Trans. R. Soc. Lond. A 1991 **337**, 429-466
doi: 10.1098/rsta.1991.0133

Stability of forced steady solitary waves

By R. Camassa† and T. Yao-tsu Wu

Division of Engineering and Applied Science, California Institute of Technology, Pasadena, California 91125, U.S.A.

Contents

	PAGE
1. Introduction	430
2. Forced solitary waves	432
3. The stability of forced solitary waves	433
4. Linear stability analysis	435
(*a*) *A perturbation expansion for the eigenvalues*	436
(*b*) *Global spectral behaviour*	438
5. Nonlinear stability	442
6. Existence of multiple stationary solutions	445
7. Numerical simulations	448
(*a*) *The periodical bifurcating (transcritical) régime*	449
(*b*) *The aperiodical bifurcating régime*	453
(*c*) *The stable supercritical régime*	455
8. Conclusions	456
Appendix A. Local spectral analysis	457
(*a*) *The inner problem for the case of* $m = 2$, $\mu_m = m^2 = 4$	458
(*b*) *The outer problem for the case of* $\mu = 4$	459
(*c*) *Inner and outer expansions for* $\mu = 1, 9$ *and other cases*	460
Appendix B. Global spectral analysis	461
Appendix C. A hamiltonian system and its conservation laws	463
References	465

This paper explores the basic mechanism underlying the remarkable phenomenon that a forcing excitation stationary in character and sustained at near resonance in a shallow channel of uniform water depth generates a non-stationary response in the form of a sequential upstream emission of solitary waves. Adopting the forced Korteweg–de Vries (fKdV) model and using two of its steady forced solitary wave solutions as primary flows, the stability of these two transcritical steady motions is investigated, and their bifurcation diagrams relating these solutions to other stationary solutions determined, with the forcing held fixed. The corresponding forcing functions are characterized by a velocity parameter for one, and an amplitude parameter for the other of the steadily moving excitations.

† Present address: Los Alamos National Laboratory, Los Alamos, New Mexico 87545, U.S.A.

Phil. Trans. R. Soc. Lond. A (1991) **337**, 429–466
Printed in Great Britain

The linear stability analysis is first pursued for small arbitrary perturbations of the primary flow, leading to a singular, non-self-adjoint eigenvalue problem, which is solved by applying techniques of matched asymptotic expansions with suitable multiscales for singular perturbations, about the isolated bifurcation points of the parametric space pertaining to the stationary perturbations. The eigenvalues and eigenfunctions are then obtained for the full range of the parameters by numerical continuation of the eigenvalues branching off from the stationary-perturbation solutions that were determined by the local analysis. A highly accurate numerical scheme is developed as required for this purpose.

The linear stability analysis identifies three categories of evolution of infinitesimal disturbances superimposed to the steady state; they occur in three different parametric régimes. The first, called periodical bifurcating régime, is characterized by complex eigenvalues, with a real part much smaller than the imaginary part, signifying that small departures from the steady state will oscillate with an amplitude growing at a slow exponential rate. In the second régime, called the aperiodical bifurcating régime, the eigenvalues are purely real, implying that small departures from the steady state grow exponentially. For the third régime, linear stability theory is unable to find any eigenvalue (including zero) to exist. In this last case, however, a nonlinear analysis based on the functional hamiltonian formulation is possible, with the hamiltonian conserved for forcings of constant velocity, and the steady state is shown to be stable. For this reason, this régime will be called the stable supercritical régime.

Finally, extensive numerical simulations using various finite difference schemes are carried out to find how the solution evolves once the instability of the solution manifests, with results fully confirming the predictions obtained analytically for the various régimes. The numerical simulations show that the instability in the periodical bifurcating régime, for the type of forcings considered, causes the steady solutions to evolve into the phenomenon of periodical production of upstream-advancing solitary waves.

---

1. Introduction

Considerable attention has been paid by recent studies to the phenomenon of nonlinear, dispersive waves generated in a soliton-bearing physical system by a moving forcing disturbance sustained at resonance. A remarkable feature of the phenomenon is that a range of physical parameters exists in which a forcing excitation moving steadily with a near critical velocity in a water channel of finite depth generates a non-stationary response in the form of a sequential production of upstream-advancing solitary waves, in a process that may continue indefinitely. An example is given by the effects of a submerged topography or a surface pressure moving with a constant transcritical velocity over the top free surface of a water layer of uniform depth. Analogous phenomena can be expected to occur in various other soliton-supporting systems. A review of history and literature can be found in Wu (1987) and Lee *et al.* (1989).

A few theoretical models have been proposed for the description of this general class of motion. One of them is the generalized Boussinesq (gB) equation introduced by Wu (1979, 1981), which is applicable to wave generation and propagation by three-dimensional forcing distributions moving in arbitrary manner through a medium which may vary gradually and slowly in two horizontal dimensions. This model was first applied by Wu & Wu (1982) to predict the periodic production of solitary waves by steadily moving two-dimensional forcing disturbances sustained at resonance and later by Wu & Wu (1988) for some three-dimensional disturbances. Another approach is based on the director-sheet model of Green and Naghdi, as adopted by Ertekin *et al.* (1984, 1986) to calculate forced generations of solitary waves. The most appealing theoretical model is perhaps the forced Korteweg de Vries (fKdV) equation which characterizes unidirectional, weakly nonlinear and weakly dispersive long waves being weakly forced at resonance. Because of its simplicity in structure, this model has been used by Akylas (1984), Cole (1985) and Malomed (1988) for the point forcing and by Lee (1985), Lee *et al.* (1988, 1989) for distributed forcings. Comparisons between theory and experiment have been carefully examined by Lee *et al.* (1989), with results showing a broad agreement between experiment and various physical models in spite of some refined differences between these models. In addition, the roles played by the nonlinear and dispersive effects during the periodic generation of upstream-advancing waves can be more directly evaluated by integration with mass and energy considerations based on the fKdV model, as illustrated by Wu (1985), Grimshaw & Smyth (1986), Wu (1987) and by Lee *et al.* (1989). The relative importance of the nonlinear and dispersive effects has been investigated by Kevorkian & Yu (1989) using the Boussinesq approximation for an isolated rigid bump held fixed in an otherwise uniform stream in a rectangular channel of uniform depth. Their study includes the limiting case of nonlinear non-dispersive wave theory and the case of time-dependent Froude number. The same phenomenon has also been found to occur in a uniform channel of rectangular or arbitrary cross-sectional shape when an obstacle moves along the channel with a transcritical velocity (Teng 1990), like a boat moving in a shallow canal. However, the basic mechanism underlying the manifestation of the periodically produced, upstream-advancing waves remains so far unexplained.

The main object of this study is to explore the basic mechanism in question. It is pursued here by first considering the stability of two primary flows which are simple solutions of the forced-steady-solitary-wave family found by Wu (1987) for the fKdV model, equation (2.1) of §2. The first, case (i), given by (2.4) below, is characterized by a speed parameter pertaining to the steady transcritical velocity of translation of the forcing function, and the other, case (ii), given by (2.5) below, is controlled by an amplitude parameter characterizing the strength of a supercritical forcing distribution. Both solutions contain the classical free solitary wave of the KdV family as a special limit to which they reduce as the forcing vanishes. The second solution was proposed by Patoine & Warn (1982) to simulate a meteorological phenomenon.

In §3, a stability theory is formulated for the two forced steady solitary waves as primary flows, and a simple *a priori* estimate of the rate of growth of instabilities based on energy–momentum balance are provided. The linear stability analysis is then carried out. An eigenvalue problem is obtained by separation of variables for the fKdV equation linearized around the primary flows, as discussed in §4. The difficulty of this eigenvalue problem is partly due to the fact that the governing ordinary differential equation is of the third order and non-self-adjoint, and the analysis of these problems has remained a subject not well developed mathematically. The main objective of this section is to understand the structure of the (discrete) eigenvalue spectrum and to describe how this varies with respect to the parameters, i.e. the velocity and strength of the forcing. This is achieved by using a perturbation approach in a neighbourhood of isolated points in the parameter space where the eigenfunctions corresponding to zero eigenvalues can be computed in closed form. In this way several 'stems' of branches of eigenvalues are determined and the global structure of the spectrum is then constructed by numerical continuation over the range of parametric values in which this is possible.

There exist régimes, or intervals of parameter values, in which no eigenvalue can be found by the methods used in §4. Physically speaking, these régimes correspond to the situation in which the corresponding forcings are sufficiently 'weak' or their speed is highly supercritical. In these régimes, however, we are able to utilize the hamiltonian property of the system and show in §5 that the steady states in both cases (i) and (ii) satisfy a sufficiency criterion implying stability on nonlinear theory.

Further study along this direction leads to the determination of bifurcation points of the primary motions for forcings of type (i) to other stationary solutions; this is first shown in §6 for a neighbourhood of special parameter values (the ones identified by the linear analysis), where by using regular perturbation expansions we obtain a new stable stationary solution.

Finally, detailed features of the transient solutions through bifurcation are illustrated in §7, where extensive numerical simulations and explorations for the various cases specified by the stability analysis are performed. The results of our numerical simulations are found to confirm fully the predictions obtained analytically for the various régimes explored.

2. Forced solitary waves

In long-wave theory, the fKdV equation is particularly suitable for describing weakly nonlinear, weakly dispersive and weakly forced waves in a shallow water of uniform depth and can be written as

$$\zeta_t + (F-1)\,\zeta_x - \tfrac{3}{2}\zeta\zeta_x - \tfrac{1}{6}\zeta_{xxx} = P_x, \tag{2.1}$$

where $\zeta(x,t)$ is the free surface elevation of the water layer and the external forcing $P(x,t)$ is given by the sum of the applied surface pressure distribution and bottom topography (see Wu 1987; Lee 1985), which may be an arbitrary function of (x,t) but here will be a localized smooth function of x only, though may be impulsively started. Here $x \in \mathbb{R}$, $t \in (0, +\infty)$ and subscripts denote partial differentiation. Equation (2.1) is in non-dimensional form in which the length, time and pressure are scaled by h, $\surd(h/g)$, and ρgh, respectively, with h, g, ρ being the undisturbed water depth, gravity acceleration and uniform fluid density, respectively. The reference frame used for (2.1) is fixed with the steadily moving disturbance, in which the fluid is moving to the right with uniform constant velocity U at $x = -\infty$, corresponding to the Froude number $F = U/\surd(gh)$, and the forcing distribution is fixed for $t > 0$. The ranges of the physical parameters in which the model is derived (see Lee 1985; Wu 1987) satisfy the following estimates

$$\bar{\epsilon} \equiv (h/\lambda)^2 \ll 1, \quad a/h = O(\bar{\epsilon}), \quad |P| = O(\bar{\epsilon}^2), \quad |F-1| = O(\bar{\epsilon}), \tag{2.2}$$

where λ is a typical wavelength, and a is a typical wave amplitude. The second equation in (2.2) represents the condition that the dispersive and nonlinear effects are so properly balanced that the phenomena can be described by the present theory.

The stationary solutions of (2.1) that vanish at infinity satisfy the equation

$$(F-1)\,\zeta-\tfrac{3}{4}\zeta^2-\tfrac{1}{6}\zeta_{xx}=P(x), \tag{2.3}$$

which is obtained from (2.1) by dropping the time derivative and integrating it once in x. Among the possible solutions of (2.3) for various choices of the forcing P, soliton-like solutions ζ_s of the form

$$\zeta(x,t)=\zeta_s(x)=a\,\mathrm{sech}^2\,(kx)$$

are of particular interest here, and they occur with the forcing term assuming a $\mathrm{sech}^2\,(kx)$ or a $\mathrm{sech}^4\,(kx)$ distribution such that (Wu 1987):

(i)
$$\left.\begin{aligned}\zeta_s(x)&=\tfrac{4}{3}k^2\,\mathrm{sech}^2\,(kx),\\ P(x)&=\tfrac{4}{3}k^2(F-1-\tfrac{2}{3}k^2)\,\mathrm{sech}^2\,(kx);\end{aligned}\right\} \tag{2.4}$$

(ii)
$$\left.\begin{aligned}\zeta_s(x)&=a\,\mathrm{sech}^2\,(kx),\\ P(x)&=a(k^2-\tfrac{3}{4}a)\,\mathrm{sech}^4\,(kx),\quad F-1=\tfrac{2}{3}k^2.\end{aligned}\right\} \tag{2.5}$$

In these equations, the Froude number F in (2.4) and amplitude a in (2.5) can be regarded as a free parameter for case (i) and (ii) respectively, in addition to the parameter k which scales the length of the disturbance. We note that for case (i) the flow can be either supercritical ($F>1$) or subcritical ($F<1$); the forcing amplitude, b_1 say, is $b_1<0$ or >0 according as $(F-1)<$ or $>\tfrac{2}{3}k^2$, while the resulting $\mathrm{sech}^2\,(x)$-wave always remains positive in polarity. For case (ii), the flow can only be supercritical; the forcing amplitude reaches its maximum of $\tfrac{1}{3}k^4$ at the wave amplitude of $a=\tfrac{2}{3}k^2$, and for forcing amplitudes below this maximum, there exist two branches of wave amplitudes a for a given forcing. Nevertheless, the solution (i), or (ii), is a unique function of F in case (i), or of a in case (ii), and this will be so regarded here. Finally, we note that both solutions reduce, with the forcing vanishing, to the free soliton solution

$$\zeta_s=\tfrac{4}{3}k^2\,\mathrm{sech}^2\,(kx),\quad F-1=\tfrac{2}{3}k^2. \tag{2.6}$$

By the scaling rule (2.2), k should be a small quantity of order $O(\epsilon^{\frac{1}{2}})$. Solutions (2.4) and (2.5) are among the family of forced steady solitary waves found by Wu (1987), while solution (2.5) was originally reported by Patoine & Warn (1982).

According to the well-known uniqueness property for the initial value problem of the KdV equation on the real line of x, these steady solitary waves will be unique solutions of the fKdV equation (2.1), each of which may remain permanent in shape provided

$$\zeta(x,0)=\zeta_s(x). \tag{2.7}$$

The question of whether or not these solutions will also have a physical significance when being perturbed is closely related to their stability properties, and this is the central problem to be investigated next.

3. The stability of forced solitary waves

Equation (2.1) can be cast in homogeneous form for any perturbation $\eta(x,t)$ of the stationary solutions ζ_s corresponding to the $P(x)$ of (2.4) or (2.5) above, so that

$$\zeta(x,t)=\zeta_s(x)+\eta(x,t), \tag{3.1}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

then by (2.1) η satisfies the nonlinear evolution equation,

$$\eta_t + (\partial/\partial x)\,[(F-1)\,\eta - \tfrac{3}{4}\eta^2 - \tfrac{1}{6}\eta_{xx} - \tfrac{3}{2}\zeta_s\,\eta] = 0. \tag{3.2}$$

In terms of the similarity variables,

$$x' \equiv kx, \quad t' \equiv \tfrac{1}{6}k^3 t, \quad \eta' \equiv k^{-2}\eta, \quad F'-1 \equiv k^{-2}(F-1), \tag{3.3}$$

the parameter k is eliminated from equation (3.2), giving, after omitting the primes,

$$\eta_t + (\partial/\partial x)\,[(\mu - \alpha\,\mathrm{sech}^2\,(x))\,\eta - \tfrac{9}{2}\eta^2 - \eta_{xx}] = 0, \tag{3.4}$$

where $\mu \equiv 6(F'-1)$ and $\alpha = 12$ for forcing (i) (3.5)

and $\mu = 4$ and $\alpha \equiv 9a/k^2$ for forcing (ii). (3.6)

Since k is of $O(\epsilon^{\frac{1}{2}})$, the free parameters μ and α are of $O(1)$ and can take on values in a relatively broad range even with F and a being so constrained by (2.2).

The perturbed motion governed by (3.4) has two leading-order conservation laws. The first is the invariance of the excess mass m,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 0, \quad m \equiv \int_{-\infty}^{+\infty} \eta(x,t)\,\mathrm{d}x,$$

which is the first integral of (3.4) with respect to x under the regularity conditions at infinity. Therefore,

$$m = \text{const.} = m_0, \quad m_0 = \int_{-\infty}^{+\infty} \eta(x,0)\,\mathrm{d}x, \tag{3.7}$$

m_0 being the initial excess mass. The second relation is for the 'energy' conservation,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \alpha\int_{-\infty}^{+\infty} \eta^2 \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{sech}^2\,(x)\,\mathrm{d}x, \quad E \equiv \int_{-\infty}^{+\infty} \eta^2\,\mathrm{d}x, \tag{3.8}$$

which is the first integral of the product of (3.4) with η. Expressing $\eta = \eta_e + \eta_0$, $\eta_e(x,t)$ and $\eta_0(x,t)$ being even and odd functions of x, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} (\eta_e^2 + \eta_0^2)\,\mathrm{d}x = 2\alpha\int_{-\infty}^{+\infty} \eta_e\,\eta_0 \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{sech}^2\,(x)\,\mathrm{d}x \equiv \dot{W}, \tag{3.9}$$

which signifies that the 'total energy' pertaining to the motion η increases at the rate $\dot{W}$ which is related to the rate of working by the external forcing on the system. A sufficient condition for E to be invariant, or $\dot{W} = 0$, is that η is purely even or purely odd. In other words, E may vary only when η has both even and odd components. In such cases, the maximum rate of energy growth is readily found from (3.8) as

$$\mathrm{d}E/\mathrm{d}t \leqslant \beta E, \quad \text{or} \quad E(t) \leqslant E(0)\,\mathrm{e}^{\beta t}, \quad \beta = 4\alpha/3\surd 3,$$

β being the maximum rate of growth for $E(t)$, equal to the maximum of $\alpha\,(\mathrm{d}/\mathrm{d}x)\,\mathrm{sech}^2\,(x)$ over the real x. Thus, the energy $E(t)$ can never grow at a rate faster than β.

In the following sections, we often use another parameter, $\mathscr{A}$, defined to be the initial amplitude of $\zeta(x,0)$, i.e. $\mathscr{A} = \zeta(0,0)$ (assumed to be the extremum variation of $\zeta(x,0)$), as a measure of the perturbation strength $\eta(x,0)$. In terms of this parameter one can study the boundary of the 'basin of stability' in the $(\mu, \mathscr{A})$-space for forcings of type (i), or in the $(\alpha, \mathscr{A})$-space for forcing (ii).

4. Linear stability analysis

Linear stability analysis of the nonlinear system (3.4)–(3.6) is of significance since a state of growth or decay of $\eta(x, t)$ with a non-zero rate evaluated on linear theory cannot be altered even when additional nonlinear effects are taken into account, at least for as long as η remains small. The stability of the forced solitary waves ζ_s versus small perturbation can be determined by the linearized form of equation (3.4),

$$\eta_t + (\partial/\partial x)\,[\mu\eta - \eta_{xx} - \alpha\eta\,\mathrm{sech}^2\,(x)] = 0. \tag{4.1}$$

By introducing the separation of variables

$$\eta(x, t) = \mathrm{e}^{\sigma t} f(x), \tag{4.2}$$

where the constant σ and the function f may be complex, the real part of η being understood for physical interpretation, we obtain the following equation for $f(x)$

$$\mathrm{d}/\mathrm{d}x\,[\mathrm{d}^2 f/\mathrm{d}x^2 + (\alpha\,\mathrm{sech}^2\,(x) - \mu) f] = \sigma f, \tag{4.3}$$

or, in operator form

$$\mathscr{L}_{\mu,\alpha}(x) f = \sigma f, \tag{4.4}$$

where

$$\mathscr{L}_{\mu,\alpha}(x) \equiv \mathrm{d}/\mathrm{d}x\,[\mathrm{d}^2/\mathrm{d}x^2 + (\alpha\,\mathrm{sech}^2\,(x) - \mu)], \tag{4.5}$$

and the boundary conditions for f will be taken to be regular at infinity, i.e.

$$f^{(n)}(x) \xrightarrow[|x|\to\infty]{} 0, \quad \text{exponentially fast, for at least } n = 0, 1, 2, \tag{4.6}$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$. We note that $\mathscr{L}_{\mu,\alpha}$ is a non-self-adjoint operator. Equations (4.4–4.6) constitute an eigenvalue problem in σ, and the stability is determined by the signature of the real part of the eigenvalues, $\mathrm{Re}\,\sigma(\mu, \alpha) = 0$ giving the boundary of neutral stability in the (μ, α) space. We remark that when the stationary solution is neutrally stable, no statement can be made regarding the nonlinear stability of the perturbed waves.

This eigenvalue problem has several features of basic interest. First, the symmetries possessed by the differential operator in (4.5), namely,

$$\mathscr{L}_{\mu,\alpha}(-x) = -\mathscr{L}_{\mu,\alpha}(x), \quad [\mathscr{L}_{\mu,\alpha}(x)]^* = \mathscr{L}_{\mu,\alpha}(x), \tag{4.7}$$

where $(\cdot)^*$ denotes the complex conjugation, imply that if (4.4) has an eigenvalue σ and an eigenfunction $f(x)$, it must also possess the eigenvalues $-\sigma$ and $\pm\sigma^*$, with their corresponding eigenfunctions $f(-x)$ and $f^*(\pm x)$, respectively. Therefore we have instability for η whenever an eigenvalue with non-vanishing real part exists. Second, the regularity conditions specified in (4.6) imply, upon integration of (4.3), that

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 0 \quad \text{unless} \quad \sigma = 0, \tag{4.8}$$

that is, for non-zero eigenvalues, the excess mass of the forced solitary waves cannot be changed by perturbations of the form (4.2). This integral condition also shows that the proper eigenfunctions of (4.4) cannot constitute a basis in the linear space of C^3 functions that satisfy the regularity conditions (4.6) at infinity.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

We define the inner product (f, g) between two functions $f(x)$ and $g(x)$ by

$$(f, g) = \int_{-\infty}^{+\infty} f(x)\, g^*(x)\, \mathrm{d}x. \tag{4.9}$$

Two functions f and g are said to be orthogonal if $(f, g) = 0$. By integration by parts of the inner product $((\mathscr{L}_{\mu,\alpha} - \sigma) f, g)$, in which the terms $\mathrm{D}^i f(x)\, \mathrm{D}^j g^*(x)$, $i, j = 0, 1, 2$, where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}x$, are assumed to vanish for $x \to \infty$, we find this product equal to $(f, (\mathscr{L}^{\dagger}_{\mu,\alpha} - \sigma^*)\, g)$, and thus obtain the adjoint equation of (4.4) as

$$\mathscr{L}^{\dagger}_{\mu,\alpha}(x)\, g = \sigma^* g, \tag{4.10}$$

where
$$\mathscr{L}^{\dagger}_{\mu,\alpha}(x) \equiv -[\mathrm{d}^2/\mathrm{d}x^2 + (\alpha \operatorname{sech}^2(x) - \mu)]\, \mathrm{d}/\mathrm{d}x. \tag{4.11}$$

We note that $\mathrm{D}\mathscr{L}^{\dagger}_{\mu,\alpha} = -\mathscr{L}_{\mu,\alpha}\mathrm{D}$. Hence if $f(x)$ is an eigenfunction of (4.4) with eigenvalue σ, then $g(x)$, defined as the integral of $f^*(-x)$,

$$g(x) = \int f^*(-x)\, \mathrm{d}x, \tag{4.12}$$

is a solution of the adjoint equation (4.10) with σ^* its corresponding eigenvalue; g is called the adjoint eigenfunction of f. Further, it can be shown that f and its adjoint function g are in general not orthogonal to each other, i.e. $(f, g) \neq 0$, but

$$(\sigma - \sigma^*)\, (f, g^*) = 0, \tag{4.13}$$

of which the implications are self-evident.

The stationary solutions of (4.1) (corresponding to $\sigma = 0$) play a significant role in the present stability analysis; they can be determined explicitly since equation (4.4) can be integrated once, reducing the problem to that of a Schrödinger equation with a $\operatorname{sech}^2(x)$-potential:

$$-(K_\alpha + \mu) f \equiv \mathrm{d}^2 f/\mathrm{d}x^2 + [\alpha \operatorname{sech}^2(x) - \mu] f(x) = 0. \tag{4.14}$$

Such eigenvalue problems are, of course, quite familiar in quantum mechanics (see Landau & Lifshitz 1958, §21). In general, solutions to this equation which decay at infinity only exist for a discrete set of parameter values μ and α, say $\{\mu_m\}$ and $\{\alpha_\nu\}$, for m, ν belonging to some interval of integers.

It is possible to take advantage of the knowledge of the eigenfunctions corresponding to zero eigenvalues. First, we introduce a small parameter, ϵ, measuring the distance of μ or α from the special values $\{\mu_m\}$ or $\{\alpha_\nu\}$, and then seek an expression for the eigenvalue σ (and the corresponding eigenfunction) as an asymptotic series in ϵ. In this way the initial portion of branches $\sigma(\mu, \alpha)$ emanating from $\sigma = 0$ at μ_m and α_ν can be determined. This approach is outlined in the next section, with most of the details reported in Appendix A. Further, we notice that the eigenvalue problem (4.3) can be cast in a form which facilitates numerical computations with a high degree of accuracy, and the branches determined by the perturbation approach can be continued numerically, as illustrated in §4*b* and Appendix B.

(*a*) *A perturbation expansion for the eigenvalues*

For the definiteness, we choose to work with the forcing (i) case (i.e. set $\alpha = 12$, see (3.5)). The analysis pertaining to case (ii) is similar and the main results are reported in Appendix A. As described in Appendix A, the special parameter values $\{\mu_m\}$ at which (4.4) admits an eigenvalue $\sigma = 0$ are in this case ($\alpha = 12$) given by $\mu_m = m^2 = 1, 4$ and 9. The corresponding eigenfunctions (i.e. solutions of (4.14) regular at infinity) are

$$\mu_1 = 1, \quad f_0(x;\mu_1) = \operatorname{sech}(x)\,(5\tanh^2(x)-1), \tag{4.15}$$

$$\mu_2 = 4, \quad f_0(x;\mu_2) = \operatorname{sech}^2(x)\tanh(x), \tag{4.16}$$

$$\mu_3 = 9, \quad f_0(x;\mu_3) = \operatorname{sech}^3(x), \tag{4.17}$$

up to an arbitrary multiplicative constant. Since the eigenvalue σ is zero, these eigenfunctions need not satisfy the integral condition (4.8).

Assuming that the spectrum depends continuously on the parameters, we can search for a perturbation expansion of the eigenvalues and eigenfunctions when μ is in a neighbourhood of μ_m. Defining

$$\mu = \mu_m + s\epsilon, \quad 0 < \epsilon \ll 1, \quad s = \pm 1, \quad m = 1, 2 \text{ and } 3, \tag{4.18}$$

we can rewrite (4.4) as

$$\mathscr{L}_{\mu_m} f - s\epsilon\, \mathrm{d}f/\mathrm{d}x = \sigma(\mu) f, \tag{4.19}$$

where we have dropped the subscript α as being understood to be 12. Assume

$$\sigma(\mu) = \phi_1(\epsilon)\,\sigma_1 + \phi_2(\epsilon)\,\sigma_2 + \ldots, \tag{4.20}$$

$$f(x;\mu) = f_0(x) + \psi_1(\epsilon) f_1(x) + \psi_2(\epsilon) f_2(x) + \ldots, \tag{4.21}$$

where $f_0(x)$ represents the set of $f_0(x;\mu_m)$ in (4.15)–(4.17), and $\phi_j(\epsilon)$, $\psi_j(\epsilon) \to 0$ as $\epsilon \to 0$, $\phi_{j+1} = o(\phi_j)$, etc., $j = 1, 2, \ldots$. The terms of the product σf then show that whenever $\sigma(\mu) \neq 0$ the integral condition (4.8) can only be satisfied by the f_0 of (4.16), but not by the f_0 given in (4.15) and (4.17), and therefore it can already be seen that the perturbation expansion is a singular one. Indeed, for the 'even' function cases of (4.15) and (4.17), the eigenfunction corresponding to zero eigenvalue has an excess 'mass' (which is the integral in (4.8)) different from zero whereas the perturbed one will be required to satisfy (4.8) as soon as $\sigma \neq 0$ due to the perturbation.

The details of how this difficulty can be overcome using matched asymptotics and 'two-timing' methods are reported in Appendix A. Here we simply state the result pertaining to the eigenvalue σ in a neighbourhood of the μ_ms. The case $\mu = 4 + s\epsilon$ is different from the other two, both in the scaling with respect to ϵ (i.e. the ϕs in (4.20)) and in the fact that an eigenvalue branch can be determined both on the right and on the left of $\mu = 4$. We have in fact,

$$\sigma = \epsilon^{\frac{1}{2}} \tfrac{8}{\sqrt{15}}\, s^{\frac{1}{2}} - \epsilon \tfrac{4}{15}\, s + O(\epsilon^{\frac{3}{2}}), \tag{4.22}$$

which is purely real for $\mu > 4$ ($s > 0$), of order $O(\epsilon^{\frac{1}{2}})$ in magnitude, and is complex for $\mu < 4$ ($s < 0$), with its real part being of $O(\epsilon)$ and its imaginary part of $O(\epsilon^{\frac{1}{2}})$. The non-vanishing real part of σ will imply instability of the perturbed motion η for μ in a neighbourhood of 4, but with different rates of growth for $\mu > 4$ and $\mu < 4$, as will be further discussed below.

For the other two special values $\mu = 1, 9$ we find (see Appendix A) that branches of eigenvalues emanating from $\sigma = 0$ exist only to the left of these μs and are purely real, of order $O(\epsilon)$:

$$\begin{aligned} \sigma(\mu) &= \epsilon\, \sigma_1(\mu_m) + O(\epsilon^2) \\ &= \epsilon \tfrac{128}{27} \pi^{-2} + O(\epsilon^2), \quad \text{for } \mu = 1 - \epsilon \\ &= \epsilon \tfrac{128}{15} \pi^{-2} + O(\epsilon^2), \quad \text{for } \mu = 9 - \epsilon. \end{aligned} \tag{4.23}$$

Having determined the local structure of the discrete spectrum of the operator (4.5) for forcing (i), we follow, by numerical computation, the branches of the eigenvalues thus found beyond the small-parameter neighbourhoods where the perturbation approach can no longer hold valid.

(b) Global spectral behaviour

The relative simplicity of the operator $\mathscr{L}_{\mu,\alpha}$, (4.5), makes it possible to consider values of the parameters μ and α away from the special values pertaining to the stationary motions listed above, and to find exact solutions numerically to the eigenvalue problem of (4.3) for the whole range of the parameters.

Keeping the notation compact, we consider type (i) forcing first. By setting $\alpha = 12$, and by using the transformation

$$z = \tfrac{1}{2}(1-\tanh(x)), \quad \hat{f}(z) \equiv f(x), \tag{4.24}$$

equation (4.3) becomes (dropping the 'hats' henceforth)

$$\frac{d^3f}{dz^3}+\frac{3(1-2z)}{z(1-z)}\frac{d^2f}{dz^2}+\left[\frac{10-\mu-6(1-2z)^2}{4z^2(1-z)^2}\right]\frac{df}{dz}+\left[\frac{12z(1-z)(1-2z)+\frac{1}{8}\sigma}{z^3(1-z)^3}\right]f=0. \tag{4.25}$$

This is a third-order ordinary differential equation with three regular singularities at $z = 0, 1, \infty$, and the boundary conditions (4.6) are now

$$f(0) = f(1) = 0. \tag{4.26}$$

The indicial equation at $z = 0$ is

$$\kappa^3 - \tfrac{1}{4}\mu\kappa + \tfrac{1}{8}\sigma = 0, \tag{4.27}$$

and in order to satisfy (4.26) we retain only the roots whose real part is greater than zero. If no roots are coincident or differ by an integer number, we only need to consider the case when one of them, say κ_1, has a real part greater than zero. In fact, due to the symmetry (4.7), the indicial equation at $z = 1$ is obtained from (4.24) by changing σ into $-\sigma$, so that $\kappa_j \to -\kappa_j$, where $\kappa_j, j = 1, 2, 3$ are the solutions of (4.27), for which we note that $\kappa_1+\kappa_2+\kappa_3 = 0$. A function f which satisfies (4.25) is therefore an eigenfunction for (4.3) if we can find coefficients of linear combination c_2, c_3 in such a way that it has the following behaviour in a neighbourhood of the singular points,

$$f(z) \sim \begin{cases} z^{\kappa_1} g_1(z), & z \sim 0, \\ c_2(1-z)^{-\kappa_2} h_2(1-z) + c_3(1-z)^{-\kappa_3} h_3(1-z), & z \sim 1, \end{cases} \tag{4.28}$$

where g_1 and h_2, h_3 are analytic functions in a neighbourhood of $z = 0$ and $z = 1$ respectively. Here $\{\kappa_j\}_{j=1,2,3}$ are three solutions of (4.27) with $\mathrm{Re}\,\kappa_1 > 0$, $\mathrm{Re}\,\kappa_3 < \mathrm{Re}\,\kappa_2 < 0$. (The other case with $\mathrm{Re}\,\kappa_1 > \mathrm{Re}\,\kappa_2 > 0$, $\mathrm{Re}\,\kappa_3 < 0$ is entirely analogous.) The radius of convergence of the power series for $g_1(z)$, $h_2(z)$ and $h_3(z)$ can be expected to be 1, since the only (regular) singularities of the differential equation these functions satisfy are at 0, 1. Furthermore, the coefficients of the power series are determined by a second-order difference equation with variable coefficients. Thus, we can very efficiently solve for the coefficients numerically and sum the power series up to some (large) number of terms to find very accurate values for g_1 and the hs. Requiring that the two expressions for $f(z)$ in (4.28) match up to the second derivatives at some point z between 0 and 1 results in an equation for the (complex) spectral parameter σ, which can then be solved by using a Newton–Raphson scheme to determine the eigenvalue. The details of the procedure outlined above are reported in Appendix B.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

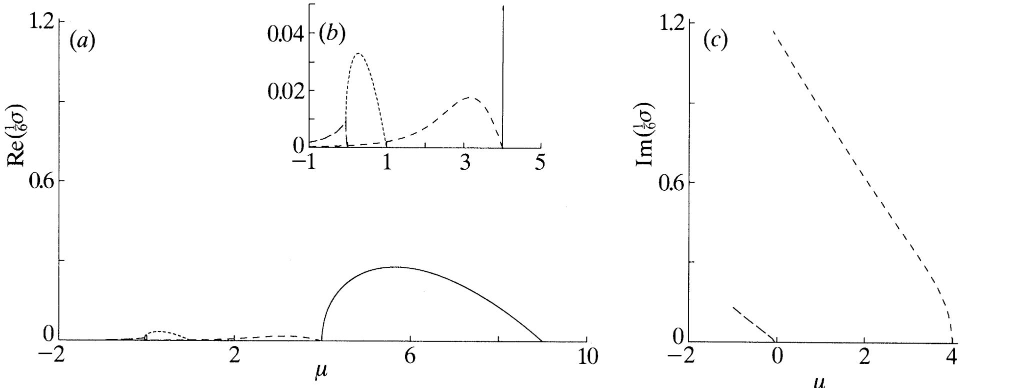


Figure 1. (a) The real part of $\frac{1}{6}\sigma$ against μ for forcing (i), as determined by the power series method of §4b. (b) A blow-up of the region close to the axis $\operatorname{Re}\sigma = 0$ for $-1 < \mu < 5$. (c) The imaginary part of $\frac{1}{6}\sigma$ against μ.

Figure 2

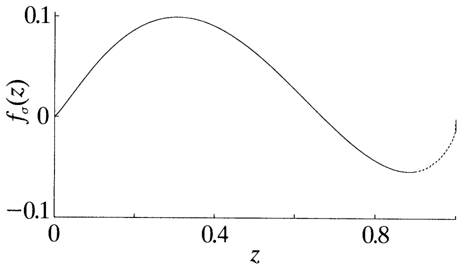
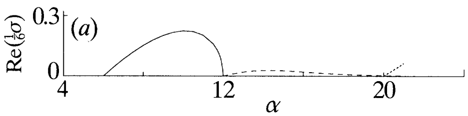


Figure 3



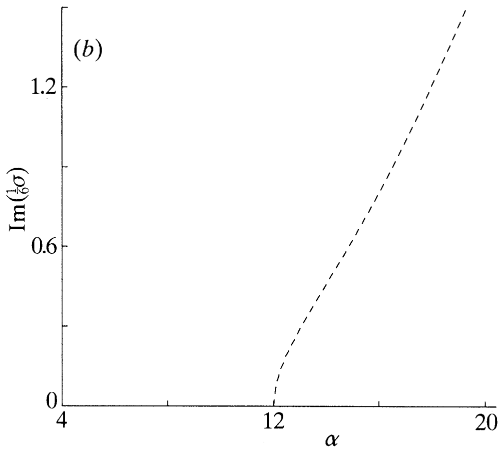


Figure 2. The eigenfunction corresponding to the eigenvalue $\frac{1}{6}\sigma = 0.2605$ at $\mu = 5$ for forcing (i), in terms of the independent variable z (equation (4.24)). The solid line is for $0 \leqslant z \leqslant 0.9$, the dashed for $0.1 \leqslant z \leqslant 1.0$.

Figure 3. (a) The real part of $\frac{1}{6}\sigma$ against α for forcing (ii), as determined by the power series method of §4b. (b) Imaginary part of $\frac{1}{6}\sigma$ against α.

The results pertaining to the eigenvalues so determined are shown in figure 1, while figure 2 depicts a typical eigenfunction for $4 < \mu < 9$. In the overlapping region $0.1 < z < 0.9$, the two expressions in (4.28) coincide up to six significant digits, when the eigenvalues are evaluated with a series truncated after 200 terms. Figure 1 shows that for the branch originating at $\mu = 4$, σ has a small real part for $\mu < 4$ and a much greater one for $4 < \mu < 9$, as expected by the previous perturbation analysis. We remark that the real part of σ remains small throughout the range $\mu < 4$, with its maximum still two orders of magnitude smaller than the imaginary part, and can only be found by high precision numerical schemes. In fact, the Galerkin expansions we tried (Wu 1988) were insufficient to provide an estimate for it. For the region $4 < \mu < 9$, however, the eigenvalue is found to be purely real as predicted by the perturbation analysis, and the branch originating at $\mu = 9$ joins with the one to the right of $\mu = 4$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The calculations for the forcing (ii) are quite similar and the results for the eigenvalues are depicted in figure 3.

Numerical experiments for the full evolution equation (2.1) have completely established the behaviour predicted by the previous spectral analysis. The inherent error of the numerical scheme should in fact play the role of a perturbation of the exact solutions (2.4) and (2.5). For (i) type forcings, the stationary solution (2.4) for $4 < \mu < 9$ is indeed unstable and decays to another stationary wave of smaller amplitude as will be shown later. The instability for $\mu < 4$ is driven by the real part of the eigenvalues, σ_r, according to (4.2), and the previous results show that for subcritical and critical cases, $\mu \leqslant 0$, σ_r is generally very small in magnitude, so small that growth of $O(1)$ from an initial ζ_s wave would require a time up to order $O(10^5)$ if the growth is seeded solely by numerical errors. However, the imaginary part should give rise to periodic oscillations in time, and this is indeed shown by the numerical results which oscillate at the period implied by the imaginary part of the eigenvalue, as will be described in detail in §7.

The special case of $\mu = 4$, $\alpha = 12$ corresponds to the free solitary wave, for which no eigenvalue different from zero can be found to exist, and the linearized analysis reduces to the one by Jeffrey & Kakutani (1972). Therefore, in the special case of free solitary waves, no definite conclusions can be drawn from linear theory and the stability problem has to be resolved by nonlinear analysis, as shown by Benjamin (1972) or by other means, including the inverse scattering formalism (Newell 1985, §3*f*).

Combining the results provided by the perturbation approach, the power series solution and the nonlinear stability analysis of the next section, we are able to predict some structure for the spectrum of the operator $\mathscr{L}_{\mu,\alpha}$ when α and μ may both vary in general, which is possible if the forcing is modified to be a linear combination of those in (2.4) and (2.5). In fact, the results obtained around the fixed-point responses at $\{\mu_m\}$, $\{\alpha_\nu\}$, corresponding to zero eigenvalues, depend mainly on the symmetries of the appropriate eigenfunctions (see Appendix A), which in turn are the solutions of the Schrödinger equation (4.14), satisfying the regularity conditions at infinity. The structure of the solutions to this latter problem, as well as for any symmetrical 'potential' with a minimum equal to $-\alpha$ at $x = 0$, and decaying fast enough at $x = \pm\infty$, is well known. Recalling that $-\mu$ plays the role of an eigenvalue in (4.14), we know that the first eigenfunction, corresponding to the lowest 'eigenvalue' of $-\mu(\alpha)$, is always symmetric about $x = 0$, the second always antisymmetric and so on in alternating fashion. The total number of 'eigenvalues' of μ will depend on α and on the rate of decay of the potential at ∞. For exponential decay, as in $-\alpha\,\mathrm{sech}^2(x)$, the number of eigenvalues is finite, and these will constitute the special points corresponding to the $\{\mu_m\}$ on which the analysis of §4*a* is based. Thus, in general, we can say that for fixed α, a μ corresponding to an even eigenfunction, μ_e say, is the starting point of a branch of purely real eigenvalues of $\mathscr{L}_{\mu,\alpha}$ for $\mu < \mu_e$. Similarly, from a $\mu = \mu_0$ corresponding to an odd eigenfunction, a branch of complex eigenvalue emanates for $\mu < \mu_0$, and a branch of real eigenvalues issues forth for $\mu > \mu_0$, the scaling of the real and imaginary parts with respect to the (small) distance $|\mu - \mu_0|$ being similar to the case of μ_2 considered above. Of course this only holds in a neighbourhood of the special points, and the numerical power series approach is needed to follow the branches as μ varies further on. We further remark that the above considerations are based on the orthogonality property of the terms of the asymptotic expansion (see Appendix A). The symmetry of the solutions of the

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

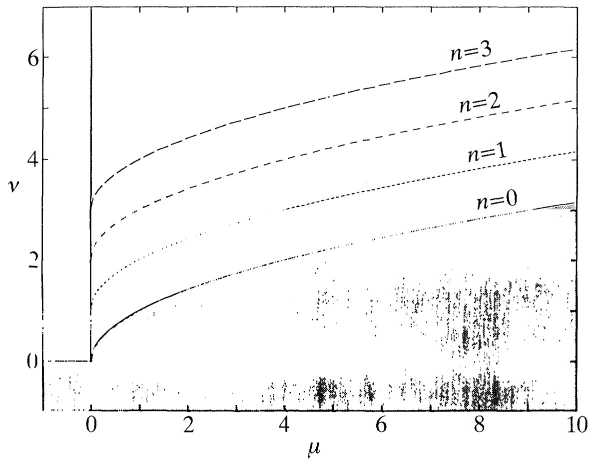


Figure 4. Subdivision of the parametric $\mu\nu(\alpha)$-plane for the existence of eigenvalues of the operator $\mathscr{L}_{\mu,\alpha}$. In the shaded region no eigenvalue with real part different from zero can exist.

Schrödinger equation (4.14) after the first eigenfunction provides only a necessary condition for these orthogonality properties to be satisfied, and modifications to the scenario outlined above might be needed in case of spurious cancellations.

In the next section, we will show that when $\mu > \bar{\mu}_e$, $\bar{\mu}_e \equiv \max\{\mu_e\}$, or $\alpha < \min \alpha_e$, the positivity of the operator $K_\alpha + \mu$ implies that no eigenvalue of $\mathscr{L}_{\mu,\alpha}$ with real part different from zero can exist. This finding agrees with the perturbation analysis which shows that no eigenvalue is found to exist in a neighbourhood of this point when $\mu > \bar{\mu}_e$. Hence the curve $\mu = \bar{\mu}_e(\alpha)$ provides a limit in the μ, α plane for the existence of eigenvalues with real part different from zero. We illustrate the above results using a $\operatorname{sech}^2(x)$ potential. It is convenient to introduce the parameter $\nu = \nu(\alpha)$ by

$$\alpha = \nu(\nu+1),$$

so that, using the same transformation as for (A 1), equation (4.14) or (A 1) has the form of a Legendre operator of order $\surd\mu$ and degree ν (Landau & Lifshitz 1958). It can then be shown that the 'eigenvalues' μ are given by

$$\nu(\alpha) - \surd\mu = n, \quad n = 0, 1, 2, \ldots, \tag{4.29}$$

their number being determined by the obvious requirement $\nu(\alpha) > n$. We note that for $\nu = n$, the solution of (4.14) is not an eigenfunction since it does not vanish at $x = \pm\infty$. The curve (a parabola in the $\mu\nu$-plane)

$$\nu(\alpha) = \surd\mu \tag{4.30}$$

for $\mu > 0$, and the line

$$\alpha = \nu(\alpha) = 0 \tag{4.31}$$

for $\mu < 0$ thus divide the $\mu\nu$-plane in two parts, the one above the curve (4.30) and line (4.31) being the region that allows existence of eigenvalues with non-zero real part. The other special points at which the eigenvalue branches described above are generated lie on the parabolae given by (4.29) with $n > 0$, see figure 4.

By using the power series approach, it can be seen that, in general, the complex eigenvalue branch originating from the 'odd' parabolae continue into the quarter plane $\mu < 0$ and $\nu > 0$, while the real branches sprouting from the 'even' parabolas merge with the real branch coming from the 'odd' parabolae in the direction of increasing μ (or decreasing α). The overall features of the dependence of the spectrum on the parameters are then similar to the cases of $\alpha = 12$, μ varying, or $\mu = 4$, α varying, as delineated above, the only difference being the position (and the number) of the points in the parametric $\mu-\alpha$ space corresponding to eigenvalues equal to zero.

5. Nonlinear stability

In the preceding Section, the linear instability for the forced solitary waves in the range $\mu < 9$, $\alpha > 6$ was established for forcing (i) and (ii) respectively, but we were unable to ascertain any eigenvalues for $\mu > 9$ or $\alpha < 6$, and so the question of stability of the forced solitary waves in the latter range of parameters is still to be resolved. A principal difficulty which is characteristic of the present problem lies in the rather unique feature of the linearized stability equation that an eigenvalue σ with a negative real part cannot be used to indicate a decay of small perturbations $f(x)$ because $-\sigma$ is then the eigenvalue for the perturbation $f(-x)$. However, a sufficiency condition for nonlinear stability can actually be shown.

We first notice that the fKdV equation (2.1) can be interpreted, as is well known for its homogeneous counterpart, as a hamiltonian system. Introducing the hamiltonian functional (see (C 13) in Appendix C)

$$\mathscr{H}(\zeta) = \frac{1}{2}\int_{-\infty}^{+\infty} [\zeta_x^2 + \mu\zeta^2 - 3\zeta^3 - 6P\zeta]\,\mathrm{d}x, \tag{5.1}$$

and the Poisson bracket

$$\{\mathscr{H}, \mathscr{G}\} \equiv \int_{-\infty}^{+\infty} \frac{\delta\mathscr{H}}{\delta\zeta}\frac{\partial}{\partial x}\frac{\delta\mathscr{G}}{\delta\zeta}\,\mathrm{d}x, \tag{5.2}$$

where $\mathscr{G}$ is any functional of ζ and $\delta/\delta\zeta$ denotes the functional derivative, we find that

$$\zeta_t = -\frac{\partial}{\partial x}\frac{\delta\mathscr{H}}{\delta\zeta}, \tag{5.3}$$

as can be readily verified by direct computation. The hamiltonian (5.1) is one of the two 'obvious' conserved quantities for the fKdV equation with the forcings under consideration, the other being the total excess mass (the equation itself being a conservation law),

$$\mathscr{M} = \int_{-\infty}^{+\infty} \zeta\,\mathrm{d}x = \text{const.} \tag{5.4}$$

These are obvious in the sense that they are the ones obtained by Noether's first theorem, due to the invariance of the lagrangian for equation (2.1) with respect to time translations and addition of an arbitrary constant to the potential for the dependent variable (see Appendix C). The presence of a non-constant forcing term has the effect of destroying the invariance with respect to spatial translations, and so the excess energy integral,

$$\mathscr{E} = \frac{1}{2}\int_{-\infty}^{+\infty} \zeta^2\,\mathrm{d}x, \tag{5.5}$$

is in general not a constant of motion.

Equation (5.3) shows that the stationary solutions $\zeta_s(x)$ of (2.1) can now be seen as extrema of the hamiltonian (5.1), i.e. they are solutions of

$$\delta\mathscr{H}/\delta\zeta = 0. \tag{5.6}$$

The second variation of the hamiltonian as ζ_s is determined as

$$\delta^2\mathscr{H} = \frac{1}{2}\int_{-\infty}^{+\infty} [\eta_x^2 - 9\zeta_s(x)\,\eta^2 + \mu\eta^2]\,\mathrm{d}x, \tag{5.7}$$

where $\eta(x,t) = \zeta(x,t) - \zeta_s(x)$ as before, and is assumed to be square integrable as well as its first derivative, i.e. it is an element of the Sobolev space H^1. Taking the time t fixed for the moment, this equation can be rewritten, in terms of the inner product (4.9) in L^2, for the usual Hilbert space of square integrable functions, as

$$\delta^2\mathscr{H} = \tfrac{1}{2}[(\eta, K_\alpha\eta) + \mu(\eta,\eta)], \tag{5.8}$$

where K_α is the Schrödinger operator introduced in (4.14), i.e.

$$K_\alpha \equiv -\mathrm{d}^2/\mathrm{d}x^2 - 9\zeta_s(x) = -\mathrm{d}^2/\mathrm{d}x^2 - \alpha\,\mathrm{sech}^2\,(x). \tag{5.9}$$

From elementary quantum mechanics, it is well known that

$$(\eta, K_\alpha\eta) \geqslant \lambda_0(\eta,\eta), \tag{5.10}$$

where $\lambda_0 = \lambda_0(\alpha)$ is the infimum of the spectrum of the operator K_α, which coincides with the (negative) eigenvalue of K_α corresponding to the ground state, if the $\mathrm{sech}^2\,(x)$-potential in it is able to support one. In our case $\lambda_0(12) = -9$ for forcing (i), while the minimum value of α for having a bound state is $\alpha = 6$ for forcing (ii), corresponding to $\lambda_0(6) = -4$. Thus

$$2\delta^2\mathscr{H} \geqslant [\mu + \lambda_0(\alpha)]\|\eta\|^2 \quad (\|\eta\|^2 = (\eta,\eta)),$$

$$\geqslant (\mu - 9)\|\eta\|^2 \quad \text{for forcing (i)}, \tag{5.11}$$

$$\geqslant [4 + \lambda_0(\alpha)]\|\eta\|^2 \quad \text{for forcing (ii)}, \tag{5.12}$$

and hence for $\mu > 9$ or $\alpha < 6$, for forcing (i) or (ii) respectively, the second variation $\delta^2\mathscr{H}$ is strictly positive, a result commonly referred to as formal stability, which is a sufficient condition for excluding linear instability (Holm *et al.* 1985). Formal stability is a step towards establishing nonlinear stability and, owing critically to the presence of the parameters μ and α, it is indeed possible to show that these forced solitary waves are (nonlinearly) stable in the range of parameters under consideration. Nonlinear stability is here intended in the usual Lyapunov sense, i.e. it is possible to find a norm $d(\cdot,\cdot)$ in the appropriate functional space on which the evolution equation (2.1) is defined, so that for any $\epsilon > 0$ one can determine a $\delta > 0$ such that if $\mathrm{d}(\zeta,\zeta_s) < \delta$ at $t = 0$, then $\mathrm{d}(\zeta,\zeta_s) < \epsilon$ at any time $t > 0$. Although a second non-trivial conserved quantity appears to be missing for the forced case, the original argument of Benjamin (1972) can be adapted to this case, and convexity estimates can be provided for the hamiltonian functional itself.

The fact that the excess energy of (2.1), $\mathscr{E}$, is no longer conserved actually simplifies the analysis, since this is a consequence of the absence of spatial translation invariance, which eliminates the need to consider quotient spaces with respect to the translations. Specifically, the total variation $\Delta\mathscr{H}$ of the functional $\mathscr{H}$ at ζ_s is

$$\Delta\mathscr{H} = \frac{1}{2}\int_{-\infty}^{+\infty} [\eta_x^2 - 9\zeta_s(x)\,\eta^2 + \mu\eta^2]\,\mathrm{d}x - \frac{3}{2}\int_{-\infty}^{+\infty} \eta^3\,\mathrm{d}x, \tag{5.13}$$

and the last term can be estimated by using the Sobolev inequality

$$\frac{3}{2}\int_{-\infty}^{+\infty}\eta^3\,\mathrm{d}x \leqslant \tfrac{3}{2}|\eta|_\infty \int_{-\infty}^{+\infty}\eta^2\,\mathrm{d}x \tag{5.14}$$

$$\leqslant \tfrac{3}{2\sqrt{2}}\,\|\eta\|_1\,\|\eta\|^2, \tag{5.15}$$

where $|\cdot|_\infty$ is the supremum norm, $\|\cdot\|_1$ is the Sobolev norm $\|\eta\|_1^2 \equiv \|\eta\|^2 + \|\eta_x\|^2$ and η is assumed to be in the corresponding space for $t > 0$. Using (5.15) it then follows that

$$\Delta\mathscr{H} \geqslant \tfrac{1}{2}[\mu+\lambda_0(\alpha)]\|\eta\|^2 - \tfrac{3}{2\sqrt{2}}\,\|\eta\|_1\,\|\eta\|^2. \tag{5.16}$$

If we take the norm d to be $\|\cdot\|_1$, and impose, at time $t = 0$,

$$\|\eta(\cdot\,,\,0)\|_1 \leqslant \delta, \tag{5.17}$$

it is easy to provide an upper bound on $\Delta\mathscr{H}$, again using Sobolev inequality (5.15),

$$\begin{aligned}\Delta\mathscr{H} &\leqslant M\|\eta\|_1^2 + \tfrac{3}{2\sqrt{2}}\,\|\eta\|_1\,\|\eta\|^2,\\ &\leqslant M\|\eta\|_1^2 + \tfrac{3}{2\sqrt{2}}\,\|\eta\|_1^3,\\ &\leqslant \gamma(\delta) \quad \text{say},\end{aligned} \tag{5.18}$$

where $M = \tfrac{1}{2}\max\,(1/|\mu+\lambda_0(\alpha)|)$. Because of the invariance of $\mathscr{H}$ with time, this bound holds for all time $t > 0$ for which $\eta(x,t)$ exists in H^1.

Following Bona (1975), introducing for brevity the notation

$$A(t) \equiv \|\eta\|, \quad B(t) \equiv \|\eta_x\|, \tag{5.19}$$

we can further provide a bound on $B(t)$ for any time $t \geqslant 0$ from the inequality (5.18), isolating the η_x term in it,

$$B^2 \leqslant 2\gamma + \tfrac{3}{\sqrt{2}}\,(A+B)\,A^2 + |\mu+\lambda_0(\alpha)|A^2. \tag{5.20}$$

Once this inequality is solved for B, we can estimate B in terms of A,

$$B \leqslant F(A), \tag{5.21}$$

where

$$F(A) = \tfrac{1}{2}\{3A^2 + [9A^4 + 8\gamma + 4(3A^3 + |\mu+\lambda_0(\alpha)|A^2)]^{\frac{1}{2}}\}. \tag{5.22}$$

Since (5.16) implies

$$\gamma \geqslant \Delta\mathscr{H} \geqslant \tfrac{1}{2}A^2\,[(\mu+\lambda_0(\alpha)) - 3/\sqrt{2}\,(A+B)], \tag{5.23}$$

using (5.21) gives

$$\gamma \geqslant \tfrac{1}{2}A^2[(\mu+\lambda_0(\alpha)) - 3/\sqrt{2}\,F(A)] - \tfrac{3}{2\sqrt{2}}A^3. \tag{5.24}$$

Now, since $F(0) = \sqrt{(2\gamma)}$, hence if

$$\tfrac{1}{3}[\mu+\lambda_0(\alpha)] > \sqrt{\gamma}, \tag{5.25}$$

there must exist some positive interval $I_\gamma \equiv [0, A_\gamma]$ such that, if A belongs to I_γ, the right hand side of (5.24) is positive, and monotonically increasing with A, as can be realized by studying it as a function of A. Therefore, with γ (and hence δ) being held fixed and sufficiently small, and taking the corresponding A, A_γ say, we have a bound on A in terms of γ, $A \leqslant A_\gamma$ and in turn, using (5.21), we also have a bound on B in terms of γ only, $B \leqslant B_\gamma$, for any $t \geqslant 0$. Hence, by choosing δ such that both (5.25) and

$$A^2 + B^2 \leqslant A_{\gamma(\delta)}^2 + B_{\gamma(\delta)}^2 \leqslant \epsilon^2 \tag{5.26}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

hold, the Lyapunov condition is established with $d = \| \, \|_1$. We again see that the condition expressed by (5.25) is crucial, and this of course is attained for $\mu > 9$ and $\alpha < 6$ for forcing (i) and (ii), respectively. It is interesting to notice that for a stationary state which is negative definite, the spectrum of the operator in (5.9) is purely continuous, $0 < \lambda < +\infty$, and such a solution would therefore be stable, according to (5.25), for as long as $\mu > 0$, or the forcing moves at supercritical speed. This proves the conjecture by Malomed (1988) for a negative solution corresponding to a forcing $P(x)$ given by a δ-function. In §7, the stability character of negative-definite solutions is demonstrated by carrying out numerical simulations on the full fKdV equation (2.1).

Underlying the above considerations is the assumption that for sufficiently smooth initial data $\zeta(x,0)$ (and forcing P in general) the solution $\zeta(x,t)$ exists, continuous in time, and belongs to at least H^1 for all time $t > 0$ (see Bona & Smith 1975).

It is interesting to notice that an almost identical argument can be provided for the nonlinear stability of the stationary solutions to the regularized fKdV equation, which is the one to be used in most of the numerical simulations of §7. In the coordinate frame and notation of (2.1), this equation can be written as

$$\zeta_t + 6(F-1)\,\zeta_x - 9\zeta\zeta_x - \zeta_{xxt} - F\zeta_{xxx} = 6P(x). \tag{5.27}$$

The analogue of (2.4) and (2.5), i.e. the stationary solutions $\zeta_{\mathrm{sr}}(kx)$, in this case are

$$(\mathrm{i_r}) \qquad \left.\begin{aligned} \zeta_{\mathrm{sr}}(x) &= \tfrac{4}{3}k^2 F\,\mathrm{sech}^2\,(kx), \\ P_{\mathrm{r}}(x) &= \tfrac{4}{3}k^2(F-1-\tfrac{2}{3}k^2F)\,\mathrm{sech}^2\,(kx), \end{aligned}\right\} \tag{5.28}$$

$$(\mathrm{ii_r}) \qquad \left.\begin{aligned} \zeta_{\mathrm{sr}}(x) &= a\,\mathrm{sech}^2\,(kx), \\ P_{\mathrm{r}}(x) &= a(k^2F-\tfrac{3}{4}a)\,\mathrm{sech}^2\,(kx), \quad F-1 \equiv \tfrac{2}{3}k^2, \end{aligned}\right\} \tag{5.29}$$

and the hamiltonian for this regularized fKdV equation only requires a change in the coefficient of the derivative term in (5.1) (see Appendix C),

$$\mathscr{H}_{\mathrm{r}} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} [F\zeta_x^2 + 6(F-1)\,\zeta^2 - 9\zeta^3 - 6P(x)\zeta]\,\mathrm{d}x. \tag{5.30}$$

Thus, introducing the notation

$$x' = kx, \quad \mu_{\mathrm{r}} \equiv 6(F-1)/k^2F, \quad \alpha_{\mathrm{r}} \equiv 9a/k^2F, \tag{5.31}$$

in analogy with (3.5) and (3.6) we can write the total variation of the hamiltonian based on $\zeta_{\mathrm{sr}}(x)$ as

$$\frac{1}{kF}\Delta\mathscr{H}_{\mathrm{r}} = \frac{1}{2}\int_{-\infty}^{+\infty} [\eta_x^2 - \alpha_{\mathrm{r}}\,\mathrm{sech}^2\,(x')\,\eta^2 + \mu\eta^2]\,\mathrm{d}x' - \frac{3}{2k^2F}\int_{-\infty}^{+\infty}\eta^3\,\mathrm{d}x', \tag{5.32}$$

and from here on the previous arguments apply virtually unchanged, with the exception of the coefficient in front of the cubic term in (5.13).

6. Existence of multiple stationary solutions

The spectrum of the operator K_α in (4.14) is once again useful for determining bifurcation points of the solutions (2.4) and (2.5) to other stationary solutions. More specifically, we consider the time-independent counterpart of equation (3.4) in a

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

neighbourhood of the parameter values $\mu = \mu_m$ (or $\alpha = \alpha_\nu$), which after integration in x under condition (4.6), can be written as

$$-\eta_{xx} - \alpha\eta\,\mathrm{sech}^2(x) + \mu\eta - \tfrac{9}{2}\eta^2 = 0. \tag{6.1}$$

We note that (6.1) has a solution of the form $\eta_s = b\,\mathrm{sech}^2(x)$ if $\mu = 4$ and $b = \frac{2}{9}(6-\alpha)$, so that for forcing (i), $\mu = 4$ ($\alpha = 12$), we have $b = -\frac{4}{3}$, making the resultant motion trivial (i.e. $\zeta = \zeta_s + \eta_s = 0$) with $P = 0$, whereas for forcing (ii), $b = 0$, $P = 0$, giving a free soliton for the resultant ζ. We proceed below to determine non-trivial solutions of (6.1) for other values of μ and α.

For definiteness, let us first restrict ourselves to forcing (i), with $\alpha = 12$ so that the subscript α of K_α may be omitted without causing ambiguity, and define μ as in (4.18), i.e.

$$\mu = \mu_m + s\epsilon, \quad 0 < \epsilon \ll 1, \quad s = \pm 1, \quad m = 1, 2, 3. \tag{6.2}$$

We seek for solutions of (6.1) by regular perturbation expansion of the form

$$\eta(x;\mu) = \psi_1(\epsilon)\,\eta_1(x) + \psi_2(\epsilon)\,\eta_2(x) + \psi_3(\epsilon)\,\eta_3(x) + \ldots, \tag{6.3}$$

where the $\{\psi_j\}$ are taken as in (A 2) and (A 3), i.e. $\psi_j(\epsilon) = \epsilon^j$ for $\mu = 1$, 9 and $\psi_j(\epsilon) = \epsilon^{j/2}$, for $\mu = 4$. Substituting these expressions in (6.1) we have for $m = 1, 3$

$$(K+\mu_m)\,\eta_1 = 0, \quad (K+\mu_m)\,\eta_n = s\eta_{n-1} + \frac{9}{2}\sum_{k=1}^{n-1} \eta_{n-k}\,\eta_k \quad (n > 1), \tag{6.4}$$

and for $m = 2$,

$$(K+4)\,\eta_1 = 0, \quad (K+4)\,\eta_2 = \tfrac{9}{2}\eta_1^2, \tag{6.5}$$

$$(K+4)\,\eta_n = s\eta_{n-2} + \frac{9}{2}\sum_{k=1}^{n-1} \eta_{n-k}\,\eta_k \quad (n > 2), \tag{6.6}$$

where K is the Schrödinger operator defined in (4.14). Thus, for the case $m = 2$, $\mu_m = 4$, we find that for the first order (in the notation of (4.15)–(4.17)),

$$\eta_1(x) = c_1 f_0(x;4) = 2c_1 \tanh(x)\,\mathrm{sech}^2(x). \tag{6.7}$$

In analogy with (A 5), the inhomogeneous equation for η_2 in (6.5) is solvable, due to the symmetries of the functions involved, and the particular solution for η_2 can be determined as

$$\eta_2(x) = -\tfrac{3}{4}c_1^2\,\mathrm{sech}^2(x)\,[3\,\mathrm{sech}^2(x) - 2\tanh(x) - 2]. \tag{6.8}$$

The constant c_1 is determined by the solvability condition at the next order, $n = 3$,

$$\frac{s}{9}\|f_0\|^2 - (f_0^2, \eta_2) = 0,$$

to give

$$c_1 = \pm\tfrac{1}{3}\surd(-11s), \tag{6.9}$$

the square root arising from the proportionality $\eta_2 \propto c_1^2$. Therefore, in order to have η_2 real we must take $s = -1$, i.e. the bifurcating solutions exists only for $\mu < 4$, and these small amplitude deviations from the exact solution ζ_s of (2.4) for $\mu < 4$ have the form, to the first order,

$$\zeta(x) = \tfrac{4}{3}\,\mathrm{sech}^2(x) \pm \tfrac{2}{3}(11\epsilon)^{\frac{1}{2}}\,\mathrm{sech}^2(x)\tanh(x) + O(\epsilon) \quad \text{at} \quad \mu = 4-\epsilon. \tag{6.10}$$

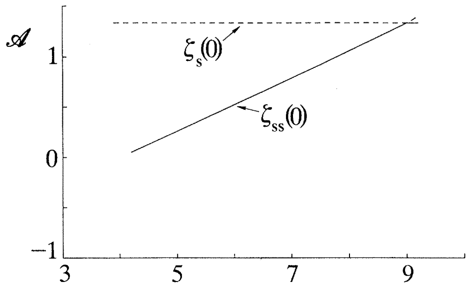


Figure 5. Bifurcation diagram pertaining to the stationary solution ζ_s and the second stationary solution ζ_{ss} for forcing (i), as obtained by numerically solving (6.14). Plotted here is the amplitude $\mathscr{A}$ of $\zeta_s(x)$ and of $\zeta_{ss}(x)$ against μ, the dashed line referring to instability, the solid to stability.

By similar calculations, the solutions bifurcating at the other two eigenvalues (corresponding to $\mu = 1$ and $\mu = 9$) of the operator K are

$$\zeta(x) = \tfrac{4}{3}\operatorname{sech}^2(x) + s\epsilon c_1(\mu_m) f_0(x;\mu_m) + O(\epsilon^2), \quad m = 1, 3, \tag{6.11}$$

where
$$c_1(\mu_m) = \frac{2}{9}\int f_0^2(x;\mu_m)\,\mathrm{d}x \Big/ \int f_0^3(x;\mu_m)\,\mathrm{d}x, \tag{6.12}$$

with the integrals ranging from $-\infty$ to $+\infty$. Thus, the amplitude constant $c_1(\mu_m)$ is

$$c_1(\mu_1) = \frac{2^{11}}{3^4\cdot 67\cdot \pi} \approx 0.120, \quad c_1(\mu_3) = \frac{2^{12}}{3^3\cdot 5^2\cdot 7\cdot \pi} \approx 0.276. \tag{6.13}$$

Furthermore, in seeking possible continuations of the new branch of solutions (6.11) to $\mu = 4$, we note that the forcing function P in (2.4) becomes of order ϵ for $\mu = 4 + s\epsilon$, as the stationary version of (2.1) shows upon using the similarity transformation (3.3),

$$-\zeta_{xx} + 4\zeta - \tfrac{9}{2}\zeta^2 + s\epsilon[\zeta - \tfrac{4}{3}\operatorname{sech}^2(x)] = 0. \tag{6.14}$$

Thus we adopt a perturbation expansion of the solution about $\zeta = 0$, $\mu = 4$, to seek a branch of solutions $\zeta(x;\epsilon)$ bifurcating from the zero solution at $\mu = 4$, of the form

$$\zeta(x;\epsilon) = \epsilon\zeta_1(x) + \epsilon^2\zeta_2(x) + \dots. \tag{6.15}$$

We find, for the first order,

$$\zeta_{1xx} - 4\zeta_1 + \tfrac{4}{3}s\operatorname{sech}^2(x) = 0, \tag{6.16}$$

and so
$$\zeta(x;\epsilon) = \tfrac{1}{3}s\epsilon\int_{-\infty}^{+\infty} \mathrm{e}^{-2|x-\xi|}\operatorname{sech}^2(\xi)\,\mathrm{d}\xi + O(\epsilon^2). \tag{6.17}$$

On the other hand, by numerical integration of (6.14) regarding ζ as even, i.e. $\zeta_x(0) = 0$ and ϵ as arbitrary, using a fourth-order Runge–Kutta integrator and bisection method for matching the zero boundary condition at large x, it is possible to see that this new branch of solutions, which will be denoted by ζ_{ss}, joins the one originated at $\mu = 9$ with wave amplitude $a = \frac{4}{3}$ (see (6.11)); the result is given in figure 5, where the amplitude $\mathscr{A}$ of ζ_{ss} and ζ_s are plotted against the parameter μ.

The stability character of the new stationary solution ζ_{ss} found by the above regular perturbation methods and the full bifurcation curve of ζ_{ss} away from the special parameter values $\{\mu_m\}$ are issues that need to be addressed. According to the nonlinear stability analysis of §5, the first question can be answered once the ground-state eigenvalue of the operator K with 'potential' $\zeta_s(x)$ given by one of these stationary solutions has been evaluated. Once again, in a neighbourhood of μ_m, this is possible by using perturbation techniques. For instance, for the case $m = 3$ we have

$$K_\epsilon \equiv \mathrm{d}^2/\mathrm{d}x^2 - 12\,\mathrm{sech}^2(x) - s\epsilon[9c_1(\mu_3)\,\mathrm{sech}^3(x)], \tag{6.18}$$

and by an elementary perturbation calculation the corresponding ground state eigenvalue is,

$$\lambda_0(12;\epsilon) = \lambda_0(12) - s\epsilon[9c_1 \textstyle\int f_0^3(x;\mu_m)\,\mathrm{d}x / \int f_0^2(x;\mu_m)\,\mathrm{d}x] + O(\epsilon^2) = -9 - 2s\epsilon + O(\epsilon^2). \tag{6.19}$$

Thus, for $s = -1$, i.e. $\mu = 9 - \epsilon$, we have

$$\mu + \lambda_0(12;\epsilon) = \epsilon > 0, \tag{6.20}$$

and so the positive lower bound estimate for the corresponding $\Delta\mathscr{H}$ is established. Under the forcing (i), the bifurcation of stationary solutions to (2.1) taking place at $\mu = 9$ is therefore of the transcritical type, and a stability exchange occurs at $\mu = 9$, $\mathscr{A} = \zeta_s(0) = \frac{4}{3}$.

7. Numerical simulations

The salient features of the forced solitary wave solutions delineated in the preceding discussion can constitute an ideal trial ground for validating any numerical code developed for solving the fKdV (2.1). In fact, if the initial condition $\zeta(x,0) = \zeta_s(x)$, equal to one of the two stationary types of forcing (i) and (ii) is chosen, an accurate code should be able to show evidence of such stability features as determined analytically in the foregoing. Since the numerical results are by nature approximations, with errors usually estimable, we would expect that for the cases where the exact solution is found to be unstable, there should arise, in due time, in the corresponding numerical results a spontaneous onset of the instability, which can be attributed solely to the ever-present numerical errors, with no need to superimpose a perturbation to the exact solution. By decreasing the truncation error of the numerical code, for instance by adopting a finer grid, the manifestation of any instability should accordingly be somewhat delayed in time. Conversely, by perturbing the initial condition to provide a departure from a stable stationary wave, the numerical results should show evidence of the tendency of the system to recover the stationary state if it is stable, and if the perturbation is not too large. Thus, the computer simulations can be helpful in describing how the system evolves out of an unstable state, a question that the stability analysis of the previous sections is insufficient to address.

We shall focus our numerical simulations for forcing (i) on the three intervals of the speed-parameter μ (see (3.5)), as identified by the previous analysis, namely $\mu < 4$, $4 < \mu < 9$ and $\mu \geqslant 9$, which are called the periodical bifurcating (transcritical) régime, aperiodical bifurcating régime, and stable supercritical régime, respectively. We further notice that for $\mu < 4$ the amplitude of the forcing given by (2.4) is negative. For forcing (ii), since it provides an analytic expression for solitary waves of negative amplitude, we have concentrated our numerical study on values of $\alpha < 0$ for the reason given in §5.

One of the codes employed here is based on the one developed by Wu & Wu (1982) which uses the modified Euler's predictor–corrector algorithm in advancing time, with the space derivatives approximated by central differences. The forward time-difference computation for $\zeta(x,t)$ is implicit, to achieve the desired numerical stability and accuracy with a relatively large time step (up to 0.2 in dimensionless form), and no iteration is required on the corrector stage. Furthermore, the open boundary conditions have been adopted at the space boundaries of the computation domain that require the waves adjacent to each boundary to leave the computation domain at the rate of the linear wave velocity $c_0 = \sqrt{(gh)}$. To avoid propagation of short wavelength disturbances that can be generated by numerical errors, equation (5.27), the so called 'regularized' fKdV, has been adopted rather then the fKdV equation itself, the two models being equivalent in the limit of applicability, i.e. for long waves (small k, see (2.6)), and small amplitude. The differences between the results obtained from the two models is expected to be of $O(k^2)$ (Benjamin *et al.* 1972; Whitham 1974; Wu 1987) and, as already mentioned, the previous stability analysis applies to this equation as well, with only minor modifications. We notice that according to the similarity transformation (3.3), the choice of taking k small has to be counterbalanced by the drawback that a smaller k implies a larger number of time steps (in physical time). In all the cases in which this code was used, the parameter k is kept fixed at a compromised value of 0.3.

Throughout the simulations the wave resistance coefficient $C_{\mathrm{R}}(t)$, defined as

$$C_{\mathrm{R}}(t) = -\int_{-\infty}^{+\infty} P(x)\,\zeta_x(x,t)\,\mathrm{d}x, \tag{7.1}$$

has been evaluated; this quantity has been found very useful to better emphasize some major features such as the presence of small-amplitude oscillations, their time period, etc. which otherwise would become difficult to detect. Physically, $C_{\mathrm{R}}(t)$ has its significance in providing a non-dimensional measure of the power being supplied by the forcing, as shown by the energy balance equation,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} \zeta^2(x,t)\,\mathrm{d}x = -\int_{-\infty}^{+\infty} P(x)\,\zeta_x(x,t)\,\mathrm{d}x, \tag{7.2}$$

which is obtained by integrating the product of (2.1) with ζ over x.

To fully appreciate the previous remarks, we also have developed a simple finite difference code to numerically solve the fKdV equation. It is based on the explicit leap-frog scheme introduced by Zabusky & Kruskal (1965), with three-point average on the nonlinear term to comply with the energy balance equation (7.2). Being explicit, the code is much less efficient than that specified above for the regularized fKdV model, and for numerical stability the typical time step Δt for $k = 0.5$ has to be $\frac{1}{50}$ the spatial step Δx, as compared with the typical case of $\Delta x = 0.1$, $\Delta t = \frac{1}{2}\Delta x$ for the first code. Therefore, application of the latter code has been limited to cases when spot comparisons with the results of the regularized fKdV equation were desired. All the plots shown here are referred to the 'body frame' in which the forcing is stationary, i.e. the water is entering the computational domain from the left side with non-dimensional velocity F.

(*a*) *The periodical bifurcating* (*transcritical*) *régime*

The results for forcing (i) within the range $\mu < 4$ can be summarized as follows.

(*a*) The phenomenon of periodic generation of the so called 'runaway' solitons as reported by Wu & Wu (1982) is invariably observed for $\mu \lesssim 2$, with the zero initial

Figure 6

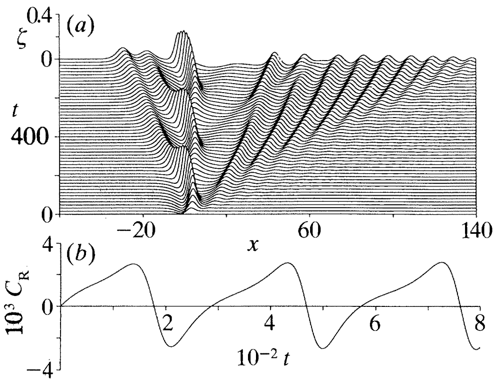


Figure 7

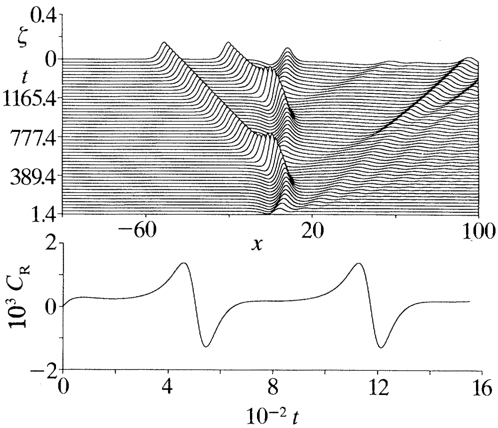


Figure 6. (*a*) The phenomenon of generation of upstream-running waves for forcing (i) at $\mu = 0$ ($F = 1$), with the zero initial condition $\zeta(x,0) = 0$ ($\eta(x,0) = -\zeta_s(x)$), simulated with $k = 0.3$, and computed with $\Delta x = 0.1$, $\Delta t = 0.05$, using the implicit code. (*b*) The corresponding wave resistance coefficient against time. Its nondimensional period of oscillation $T_r' = T_r k^3$ is *ca.* 7.9 and coincides with the period of birth of upstream-running waves.

Figure 7. (*a*) The phenomenon of generation of upstream running waves for forcing (i), at $\mu = 2$ ($F = 1.03$, $k = 0.3$), with the zero initial condition $\zeta(x,0) = 0\,(\eta(x,0) = -\zeta_s(x))$, and computed with $\Delta x = 0.1$, $\Delta t = 0.05$, using the implicit code. (*b*) The corresponding wave resistance coefficient against time. Its non-dimensional period of oscillation T_r' is ≈ 18.1 and coincides with the period of birth of upstream-running waves.

condition, i.e. the rest state $\zeta(x,0) \equiv 0$. The period of generation is found to increase monotonically as μ is increased until $\mu \approx 2.5$ where it becomes impossible to get more than one soliton generated unless the (non-dimensional) time interval of computation would be extended considerably beyond 1600, which was kept as a reasonable upper limit. This can be seen in figures 6 and 7 for $\mu = 0$ and $\mu = 2$ respectively, in which the periodic nature of soliton generation is conspicuously locked in with the evolution of the wave resistance coefficient $C_R(t)$. For the case $\mu = 0$, the non-dimensional period $T_r' = T_r k^3$ (differing from t' in (3.3) by a factor of $\frac{1}{6}$) of the wave resistance coefficient C_R is $T_r' \approx 7.9$, which coincides with the period of generation of runaway solitons and also agrees with the value determined by Wu (1987) numerically as well as by applying a mass–energy theorem.

As μ is increased to $\mu = 2$, a typical case simulated in figure 7 with the zero initial condition $\zeta(x,0) \equiv 0$, gives for $C_R(t)$ the period $T_r' = 18.1$, which also coincides with the corresponding period of birth of the upstream-advancing solitary waves. For still greater values of μ in $2 < \mu < 4$, determination of T_s by numerical means becomes too elaborate to be practical due to the ever increasing period of generation of solitons. However, in this range of $2 < \mu < 4$, the second stationary solution, ζ_{ss}, which according to the nonlinear analysis is stable for $\mu < 9$, seems to become asymptotically the terminal state in the evolution of the fluid system when a sufficient departure from ζ_s is initially afforded. This behaviour is expected to prevail readily in a neighbourhood of $\mu = 3$ since there the (positive) real part of the eigenvalue, σ_r, reaches a maximum (see figure 1*b*), giving the fastest growth to small disturbances. This is illustrated in figure 8 for $\mu = 3$, with the waves evolving under forcing (i) from the initial rest state of $\zeta \equiv 0$; the free surface is seen to soon relax to

Figure 8

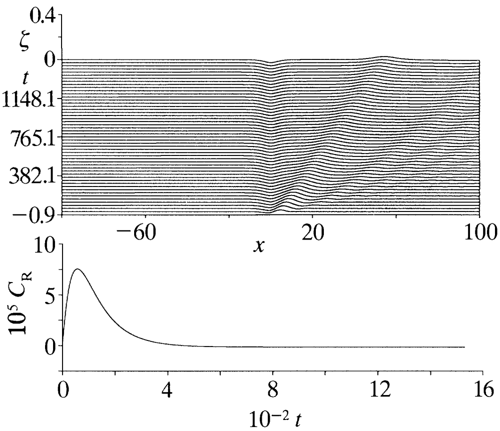


Figure 9

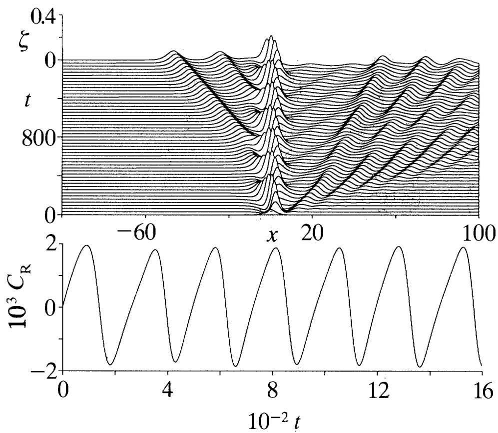


Figure 8. (*a*) Generation of the second (stable) stationary wave ζ_{ss} for forcing (i), at $\mu = 3$ ($F = 1.045$, $k = 0.3$), with the zero initial condition $\zeta(x,0) = 0\,(\eta(x,0) = -\zeta_s(x))$, and computed with $\Delta x = 0.1$, $\Delta t = 0.05$, using the implicit code. (*b*) The corresponding wave-resistance coefficient against time.

Figure 9. (*a*) The evolution of the stationary solution $\zeta_s(x)$ for forcing (i) at $\mu = 0$ ($F = 1$) showing the incipient emission of upstream-advancing solitary waves delayed with the initial condition $\zeta(x,0) = 0.4\,\zeta_s(x)$ as compared with the case shown in figure 6 for the rest initial state, the other parameters being the same. (*b*) The regular oscillation of the wave-resistance coefficient $C_R(t)$, with period $T'_r = 6.5$, is unaffected by the emission of solitary waves.

the stable ζ_{ss} state about the centre of forcing, after having radiated some initial disturbances downstream. This asymptotic state of ζ_{ss} is reaffirmed by the corresponding $C_R(t)$, which as shown in figure 8*b* has an initial positive hump signifying energy absorption from the forcing, and then settles to zero for the remaining three-quarters of the time range computed. This trend seems to continue for $3 < \mu < 4$, (but not too close to $\mu = 4$ at which $\sigma_r = 0$), especially if the rest state is chosen as the initial condition. This is physically clear since the amplitude of ζ_{ss} tends to zero as $\mu \to 4$ (see (6.15)), hence the rest state becomes merely a small perturbation.

(*b*) In the periodical bifurcating régime where the periodic upstream emission of solitary waves can occur, the nonlinear effects have a significant influence on the *incipient emission time*, T_e; the smaller the initial departure from the stationary response ζ_s, the longer the local fluctuation (incubation) will last before the first solitary wave becomes mature and emitted. After this incipient emission, the regular sequential emissions will then ensue. These salient features of the process are first illustrated in figure 9 for $\mu = 0$ ($F = 1$) with initial condition $\zeta(x,0) = 0.4\zeta_s(x)$, ($\mathscr{A} = 0.4\zeta_s(0)$). In this case, the first emission is timed at about $t = T_e = 800$ ($T'_e = T_e k^3$), near the fourth peak of $C_R(t)$, while $C_R(t)$ oscillates regularly, apparently unaffected by the emission, with period $T'_r = 6.5$, whereas the period of emission of solitons is $T'_s = 14.6$.

(*c*) If the initial condition is zero perturbation ($\eta(x,0) \equiv 0$, or $\zeta(x,0) = \zeta_s(x)$, $\mathscr{A} = \frac{4}{3}$) for μ slightly less than 4, the very slow rate of growth prevailing here for ζ and the minute departures of ζ from ζ_s (attributable only to numerical errors) makes numerical efforts prohibitive to reach the transient and terminal flow states. Nevertheless, the variations in the wave-resistance coefficient, C_R, are found to

Figure 10

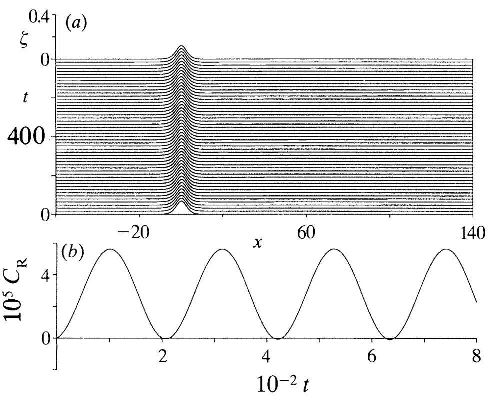


Figure 11

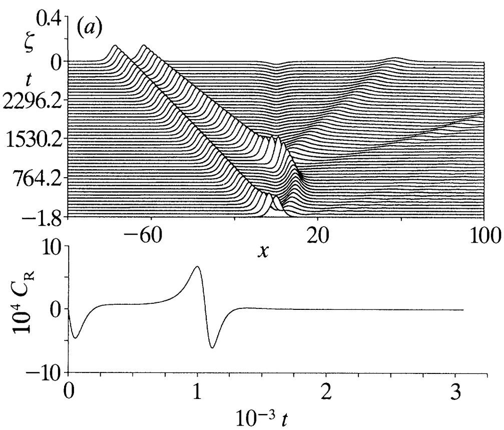


Figure 10. (*a*) The evolution of the stationary solution $\zeta_s(x)$ for forcing (i) at $\mu = 0$ ($F = 1$), with the steady-state initial condition $\zeta(x,0) = \zeta_s(x)$, computed with $k = 0.3$, $\Delta x = 0.1$, $\Delta t = 0.05$, using the implicit code. The result, with perturbation attributable solely to numerical errors, shows no emissions of solitary waves up to $t = 800$. (*b*) The corresponding wave resistance coefficient against time. The non-dimensional period of oscillation $T'_r \approx 5.75$.

Figure 11. (*a*) The evolution of the stationary solution $\zeta_s(x)$ for forcing (i) at $\mu = 3$ ($F = 1.045$, $k = 0.3$), with the initial condition $\zeta(x,0) = 1.5\zeta_s(x)$, computed with $k = 0.3$, $\Delta x = 0.1$, $\Delta t = 0.05$, using the implicit code, with the result showing upstream emissions of two solitary waves and the subsequent bifurcation to the stable solution $\zeta_{ss}(x)$. (*b*) The corresponding wave-resistance coefficient against time.

correlate very closely with the corresponding eigenvalue $\sigma = \sigma_r(\mu) + i\sigma_i(\mu)$. For example, at $\mu = 0$, figure 10*a* shows that the resulting wave has varied from $\zeta_s(x)$ merely by 2% in amplitude by the time $t = 800$, which is in accord with the estimate based on $\sigma_r(0) = 6.84 \times 10^{-4}$. With the time scaled by (3.3), it would require a time of order $O(10^5)$ at $k = 0.3$ to show an appreciable effect of $O(1)$. However, the wave resistance coefficient in figure 10*b* exhibits a regular oscillation of period

$$T'_r = k^3 T_{num} \approx 5.75, \tag{7.3}$$

though with a very small amplitude (*ca.* 5×10^{-5}).

(*d*) The nonlinear effects on the period $C_R(t)$ oscillations, T_r, compared with that on the soliton emission period, T_s, and that on the incipient emission time, T_e, are relatively weak. It has been found that the period T_r is invariably somewhat reduced with decreasing strength of emission as measured by the net variation in the wave-resistance coefficient, $\Delta C_R = C_{Rmax} - C_{Rmin}$ (which is correlated with the amplitude of the waves emitted). For the case of $\mu = 0$, we have $T'_s = 7.9, 6.5, 5.75$ against $(\Delta C_R) \times 10^4 = 53, 37, 0.55$, respectively, as shown in figures 6, 9, and 10. It is of interest to note that in the small limit of ΔC_R, the period of C_R variation becomes in good agreement with the period predicted by the imaginary part of the eigenvalue ($\omega_{ev} \equiv \mathrm{Im}\,(\frac{1}{6}\sigma) = 1.144$ at $\mu = 0$, see figure 1*b*)

$$T'_{ev} = 2\pi/\omega_{ev} = 5.494. \tag{7.4}$$

This result can be further improved by using a smaller k, or by using the original fKdV model (to replace the rfKdV equation used here), as shown by Camassa (1990).

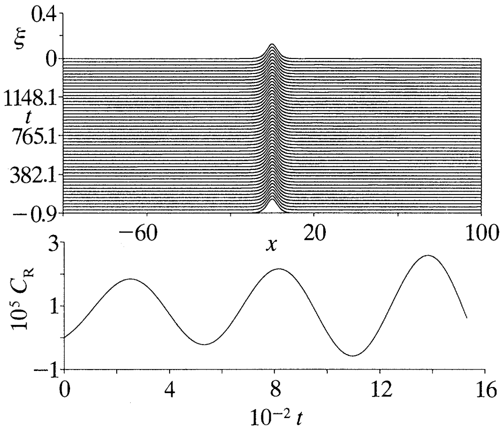


Figure 12. (*a*) The evolution of the stationary solution $\zeta_s(x)$ for forcing (i) at $\mu = 3$ ($F = 1.045$, $k = 0.3$), with the steady-state initial condition $\zeta(x,0) = \zeta_s(x)$, the numerical procedure being the same as before. (*b*) The corresponding wave-resistance coefficient $C_R(t)$ oscillates with period $T_r' \approx 15.79$ and with its amplitude growing with the factor $\exp(\gamma t')$, $\gamma = 0.125$.

Another region of interest in the periodical bifurcating régime is a subdomain centred at $\mu = 3$, near which σ_r has a maximum, albeit still quite small in value. With the initial condition $\zeta(x,0) = 1.5\zeta_s(x)$ imposed at $\mu = 3$, which is a fairly strong disturbance, figure 11 bears out the upstream emission of two solitary waves, the first being emitted in initial recoil to the forcing which is absorbing energy as indicated by C_R varying over a negative hump. Long after the first emission, the second solitary wave is emitted at about $t = 1000$, leaving the fluid system soon stably settled at the ζ_{ss} state, with C_R then falling off to zero.

Again at $\mu = 3$, but now with the zero-perturbation initial condition ($\eta(x,0) \equiv 0$, or $\zeta(x,0) = \zeta_s(x)$), the initial wave is shown in figure 12 to remain virtually unchanged up to $t = 1600$, with its numerical values exhibiting a 'breathing', or very slight oscillations with variations by less than 5% in amplitude over the duration computed. The corresponding $C_R(t)$ curve shows a growth in amplitude at a rate in accord with the real part σ_r of the eigenvalue, and its period of oscillation, evaluated numerically at

$$T_r' = k^3 T_{num} \approx 15.79, \tag{7.5}$$

compares well with that given by the imaginary part of the eigenvalue, $\omega_{ev} = \frac{1}{6}\sigma_i = 0.4$ so that

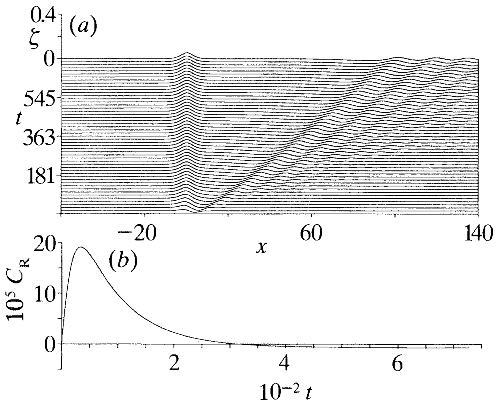
$$T_{ev}' = 2\pi/\omega_{ev} = 15.70. \tag{7.6}$$

Unfortunately, even with the local maximum rate of growth of the instability at $\mu = 3$, the computation time required to attain the asymptotic flow state is still prohibitive, with the present efficiency of our codes. The non-dimensional time for having disturbances of order $O(1)$ would in fact be, at $\mu = 3$, of the order of $O(5 \times 10^4)$.

(*b*) *The aperiodical bifurcating régime*

For μ in the range $4 < \mu < 9$, the initial state of rest is found to gradually evolve under the forcing (i) into the stable ζ_{ss} wave, after radiating a train of waves downstream. A typical case is shown in figure 13, for $\mu = 6.\overline{6}$ corresponding to $F = 1.1$ at $k = 0.3$. The asymptotic state of ζ_{ss} reached by the system is reaffirmed by the graph of C_R, figure 13*b*, indicating that no energy is being released by the forcing

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES



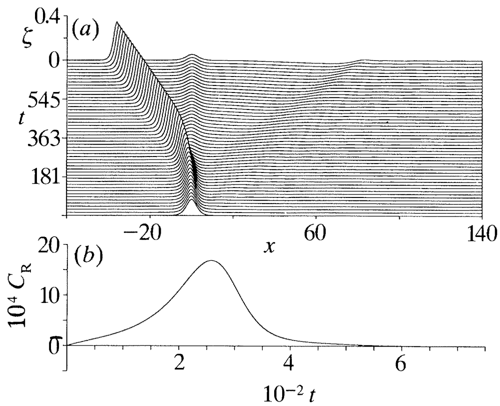


Figure 13. (*a*) The evolution of the stationary solution $\zeta_{\mathrm{s}}(x)$ for forcing (i) at $\mu = 6.\bar{6}$ ($F = 1.1$, $k = 0.3$) with the zero initial condition $\zeta(x, 0) = 0$, with the same numerical procedure as before, showing the bifurcation through transitional modes to the stable stationary wave $\zeta_{\mathrm{ss}}(x)$ after radiating downstream a train of dispersive waves. (*b*) The corresponding wave-resistance coefficient $C_{\mathrm{R}}(t)$ exhibits no periodic behaviour, its small negative asymptote being of a numerical origin.

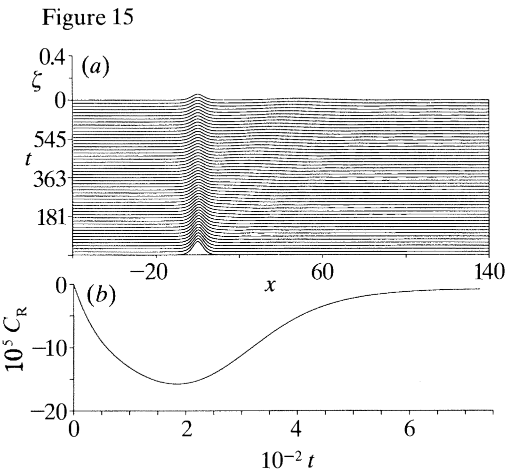
Figure 14. (*a*) A variation of the case shown in figure 13, here with the initial condition $\zeta(x, 0) = 1.1\,\zeta_{\mathrm{s}}(x)$ at $\mu = 6.6$ ($F = 1.1$, $k = 0.3$), the other parameters being the same. With this slightly stronger initial perturbation, the result shows the bifurcation of $\zeta_{\mathrm{s}}(x)$ through transitional modes to the stable state $\zeta_{\mathrm{ss}}(x)$ after the upstream emission of a single solitary wave. (*b*) The corresponding wave-resistance coefficient increases to a peak and then falls off to zero.

after the initial transient. (The negative, but small, asymptotic value reached by C_{R} is seen to be of a numerical origin, since it is halved as the spatial grid is halved, and is partly due to the finite difference scheme for the regularized fKdV model being not exactly in accord with its equivalent for the original fKdV, see Appendix C.)

This general feature of solution in the aperiodical bifurcating régime of $4 < \mu < 9$ is in sharp contrast with that in the range $\mu < 2$, see figure 6 for the case $\mu = 0$ and figure 7 for $\mu = 2$ where the periodic upstream emission of solitary waves manifests and the C_{R} exhibits a periodic behaviour in time, a trend which continues to $\mu = 3$ as reported above. This new stationary state is the stable steady wave ζ_{ss} of (2.1), as can be verified by comparing its terminal value of amplitude *ca.* $0.66\zeta_{\mathrm{s}}(0)$ (figure 13) with that of the ζ_{ss} solution at $\mu = 6.\bar{6}$, which is *ca.* $0.64\zeta_{\mathrm{s}}(0)$, on the bifurcation curve (figure 5) that bifurcates from $\mu = 9$. The instability of ζ_{s} associated with the real eigenvalue $\sigma = 0.23$ at $\mu = 6.\bar{6}$ (figure 1) manifests itself during the non-dimensional time to $t = 600$, which is about the time limit chosen for the computation and at which time the resulting wave emerges to be about fully developed.

To further pursue this point concerning the nonlinear effects, we choose to magnify the perturbation by increasing the initial wave amplitude by 10% rather than extending the time limit, and the result for $\mu = 6.\bar{6}$, $\zeta(x, 0) = 1.1\zeta_{\mathrm{s}}(x)$ is shown in figure 14. There are two remarkable features in the result: (i) the perturbed stationary wave first exhibits instability through emitting a single upstream-advancing solitary wave, and (ii) the remaining wave in a broadening region centred at the origin of forcing decays to the stable, stationary ζ_{ss} wave after emitting the upstream wave and some downstream radiation, the latter being dispersive waves moving with a subcritical speed to the right. Radiation of these waves renders C_{R} to rise in a hump before falling off to zero, signifying that the fluid system absorbs

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES



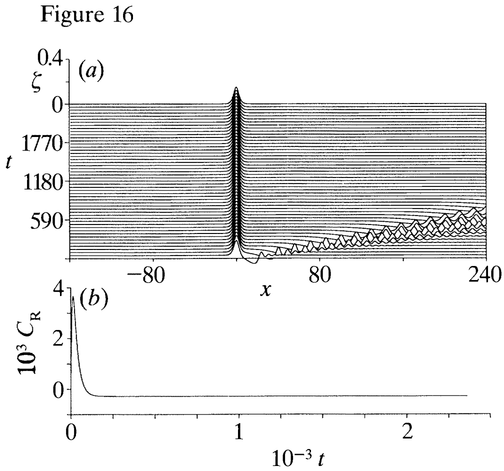


Figure 15. (*a*) Another variation of the case shown in figure 13, now with the initial condition $\zeta(x,0) = 0.9\,\zeta_s(x)$ at $\mu = 6.\bar{6}$ ($F = 1.1$, $k = 0.3$). With this slightly weaker initial perturbation, the result shows a smooth transition from $\zeta_s(x)$ to the stable state $\zeta_{ss}(x)$ with only some weak radiations downstream. (*b*) The corresponding wave-resistance coefficient indicates a slight release of energy.

Figure 16. (*a*) The generation of the stationary solution $\zeta_s(x)$ by forcing (i), at $\mu = 18$ ($F = 1.271$, $k = 0.3$), with the zero initial condition $\zeta(x,0) = 0$, computed with $\Delta x = 0.2$, $\Delta t = 0.1$, using the implicit code. The system goes to the stationary solution $\zeta_s(x)$ after a train of dispersive waves downstream. (*b*) The corresponding wave resistance coefficient against time. The system absorbs energy initially in generating the stationary wave.

energy from the forcing. On the other hand, we found that a slight decrease rather than an increase in the initial wave amplitude, implying a weaker perturbation, makes the local solitary wave decay more smoothly to the stable state ζ_{ss} while emitting only slight radiation, as can be seen in figure 15 for $\mu = 6.\bar{6}$, $\zeta(x,0) = 0.9\zeta_s(x)$, during which process a small amount of energy is released from the fluid system, as $C_R(t)$ remains negative before going to zero.

(*c*) *The stable supercritical régime*

In the stable supercritical régime of $\mu > 9$, our numerical results show evidence of their strong stability in that, even when so greatly perturbed as from the initial rest state of ζ, with $\eta(x,0) = -\zeta_s(x)$, the forced steady solitary wave ζ_s is invariably recovered after emitting some radiation downstream. The result pertaining to the case of $\mu = 18$ are graphically represented in figure 16, this result is typical of the wave evolution in this régime, persistently tending to ζ_s as the asymptotic.

With respect to the case of forcing (ii), we recall that for a specified forcing strength, there exist two branches of stationary waves $\zeta_s(x)$, one with a (similarity) amplitude $\alpha > 6$, the branch which has been shown to be unstable on linear theory (see §4*b*), and the other with an amplitude $\alpha < 6$. According to the previous analysis, with $\zeta_s(x)$ taken as a potential for the Schrödinger operator K_α, the lower branch ζ_s with $\alpha < 6$ cannot support any bound state, the spectrum being continuous and positive. It therefore follows that as a forced solitary wave this branch of ζ_s is stable and this can indeed be confirmed by numerical simulations. It is in this sense that we may regard this lower branch of $\zeta_s(x)$ as being in analogy with the steady wave $\zeta_{ss}(x)$ in stable response to forcing (i). With the stationary solution ζ_s produced by forcing (ii) so clarified, the main features of the stability of this ζ_s by numerical simulations over the stable régime $\alpha < 6$, the aperiodical bifurcating regime $6 < \alpha < 12$, and the periodical bifurcating régime $\alpha > 12$ are found in close analogy with the results that have been presented in detail for forcing (i) and will therefore be curtailed here (but see Camassa (1990) and Camassa & Wu (1991) for further details).

8. Conclusions

In summary, we have investigated the stability of stationary solutions, ζ_s, of the fKdV equation for two basic types of forcing in the class given by Wu (1987). Results obtained from the linear stability analysis play a leading role in identifying three quite distinctive transcritical régimes. The first is the aperiodical bifurcating régime characterized by a pair of purely real eigenvalues, $\sigma = \pm\sigma_r$, which exist for $4 < \mu < 9$ in case of forcing (i) and for $6 < \alpha < 12$ in case (ii) for which small departures from the stationary-response state grow exponentially. After the transient growth the original waves invariably are attracted to a new stationary wave ζ_{ss}, which is stable and is reached after some downstream radiation of dispersive waves, with or without an upstream emission of a single solitary wave to shed excessive mass and energy. The form of radiation in transition depends on the parameter $\mathscr{A}$ which is a measure of departure of arbitrarily chosen initial wave from the stationary state of ζ_s. (This measure is of course one of the myriad of means of introducing disturbances to the stationary wave ζ_s being examined, but is chosen for the convenience of computation.) Further, the evolution of the fluid system in this régime does not exhibit any time periodic phenomenon as expected in view of the eigenvalues being purely real, and indeed as the numerical simulations show.

The second régime, the periodical bifurcating régime, existing with $\mu < 4$ for forcing (i) or with $\alpha > 12$ for forcing (ii), is characterized by having a double-pair of complex eigenvalues, $\sigma(\mu) = \pm\sigma_r \pm i\sigma_i$, each pair ensuing from a fixed point of the fKdV equation, at $\mu = 4$, and a neighbourhood of $\mu = 0$ and continuing indefinitely with decreasing μ (figure 1), with the real part σ_r being two to five orders smaller in magnitude than the imaginary part σ_i. For motions in this régime, small departures from the steady response ζ_s will generally oscillate with a slowly growing amplitude. Such oscillatory growths usually give rise to periodic upstream emission of solitary waves, but may under certain circumstances undergo directly an evolution to the stable state of ζ_{ss}, such as by excessive disturbances or if μ is very close to $\mu = 4$, near which the modulus of the eigenvalue σ is very small and vanishes at $\mu = 4$. For forcing (ii), the solution ζ_s exhibits similar behaviour in the analogous régime of $6 < \alpha < 12$.

The third régime, or the stable supercritical régime, existing with $\mu > 9$ for (i) or with $\alpha < 6$ for (ii), is characterized by having no eigenvalues that can be found on linear theory. Nevertheless, the forced stationary waves in this régime are shown on a nonlinear theory to be stable in the Lyapunov sense. The numerical simulations demonstrate that the bounds estimates of the hamiltonian functional put forth in the proof for representing the system in this régime are too strict, at least with respect to the type of perturbation used (which are obtained by simply varying $\mathscr{A}$).

From the present study we may take note of the richness of new physical and mathematical contents of the general subject of nonlinear, dispersive systems sustaining forcings at resonance. The instability of the primary wave within the periodical bifurcating régime and with finite amplitude of the forcing disturbances offers a well defined route for the evolution to the régime of sequential births of upstream-radiating solitary waves in response to the steady transcritical forcings (i) or (ii). For the velocity parameter above a certain threshold ($\mu > 9$ for forcing (i)) and for the amplitude parameter below a certain margin ($\alpha < 6$ for forcing (ii)), the phenomenon of periodic generation of solitary waves ceases to manifest because the forcing is moving too fast to be out raced by any free wave in case (i) (with an amplitude within the limits of applicability of the fKdV model), or because the forcing and the resulting waves are too weak to bring the nonlinearity to effect, as in case (i) for $3 < \mu \lesssim 4$ or as in case (ii) for $\alpha < 6$. No special attention is given here separately to the subcritical forcings since the upstream radiation becomes relatively weak in this régime, though the phenomenon can persist to velocities as low as the Froude number $F = 0.2$ (Lee 1985; Lee *et al.* 1989).

In the mathematical context, the theory of eigenvalue problems seems to be still not well developed for ordinary differential equations of the third order as we have encountered in the present case. The different physical features exhibited of the phenomenon in different régimes of the parametric space are found to correlate closely with the eigenvalues being either non-existing, or purely real, or being complex conjugates. For the last case, we found that numerical methods of high accuracy are indispensable. In fact, the previous attempts by Wu (1988), using the galerkian modal expansion method up to four terms retaining the nonlinearity, and by Camassa (1990) for the linearized problem with up to 400 terms and using various basis functions, were unsuccessful to reach a definitive determination of the real component of the eigenvalue because of its minuteness as compared with the imaginary component. Such a broad disparity between the real and imaginary parts of the eigenvalue seems to be a hallmark characteristic of this class of problems that requires further investigation. From the theoretical standpoint, the variational approach expounded by Whitham (1967, 1974) and by Lighthill (1967) may be valuable for solving problems involving nonlinear dispersive waves governed by equations too complicated for analytical or numerical treatment, such as those considered here.

Finally, we note that the type of bifurcation of the solution from the specific primary wave found here seems to be new in nature. The significance and possible impact of the issues pointed out here deserves continued attention and investigation which is underway.

We take pleasure in expressing our deep gratitude to Sir James Lighthill for extremely valuable discussions, especially those concerning the case when the rate of growth of the unstable stationary motions is weak. We are indebted to George Yates for useful discussions and for his invaluable assistance in the numerical computations. This work was jointly sponsored by ONR Contract N00014-82-K-0443, NSF Grant MSM-8706045 and their successors N00014-89-J1971 and NSF 4 DMS-890 1440, the last being cosponsored by the Applied Mathematics, Computational Mathematics and Fluid Dynamics/Hydraulics Programs. One of us (R.C.) also acknowledges support by DOE Contract W-7045-ENG-36 and AFOSRISSA 900024. The numerical calculation were performed on the CRAY X-MP/48 at San Diego Supercomputer Center and at the National Center for supercomputing Applications (operated by the National Science Foundation).

Appendix A. Local spectral analysis

We report in this Appendix some of the details of the perturbation analysis for finding the eigenvalues and eigenfunctions in the forcing case (i), when μ is close to the special values $\{\mu_m\}$ where an eigenfunction corresponding to $\sigma = 0$ can be found.

Taking in (4.14) the substitution $y = \tanh(x)$ and $\mu = m^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[(1-y^2)\frac{\mathrm{d}f}{\mathrm{d}y}\right]+\left[12-\frac{m^2}{1-y^2}\right]f = 0, \tag{A 1}$$

which has solutions in the form of the associated Legendre function, $f(y) = P_3^m(y)$. Under boundary conditions that f vanishes at $y = \pm 1$, we have eigensolutions only for the discrete spectrum of $m = 1, 2, 3$, or $\mu = m^2 = 1, 4$ and 9, so that the entire set of eigenfunctions is given by (4.15)–(4.17).

We now take μ in a neighbourhood of μ_m defined with a small parameter ϵ as in (4.18), and use the expansions (4.20) and (4.21). As we have already pointed out, the perturbation problem is singular, so that the analysis has to be divided into an 'inner' and an 'outer' problem. The specific formulations and solutions of these problems are provided by the following sections.

(a) *The inner problem for the case of* $m = 2$, $\mu_m = m^2 = 4$

After some exploration, we take for the 'inner' problem

$$\phi_j(\epsilon) = \psi^j(\epsilon) = \epsilon^j, \quad \mu = 1, 9, \tag{A 2}$$

$$\phi_j(\epsilon) = \psi^j(\epsilon) = \epsilon^{j/2}, \quad \mu = 4. \tag{A 3}$$

For $\mu = 4+\epsilon s$, we have from (4.19)–(4.21) the first four order equations:

$O(1)$
$$\mathscr{L}_4 f_0 = 0, \tag{A 4}$$

$O(\epsilon^{\frac{1}{2}})$
$$\mathscr{L}_4 f_1 = \sigma_1 f_0, \tag{A 5}$$

$O(\epsilon)$
$$\mathscr{L}_4 f_2 = s\,\mathrm{d}f_0/\mathrm{d}x+\sigma_2 f_0+\sigma_1 f_1, \tag{A 6}$$

$O(\epsilon^{\frac{3}{2}})$
$$\mathscr{L}_4 f_3 = s\,\mathrm{d}f_1/\mathrm{d}x+\sigma_3 f_0+\sigma_2 f_1+\sigma_1 f_2, \tag{A 7}$$

etc. with the boundary conditions

$$f_j(x) \underset{x\to-\infty}{\longrightarrow} 0, \quad j = 0, 1, 2, \ldots. \tag{A 8}$$

The solvability condition for the above equations requires that their right-hand sides be each orthogonal to $g_0 = -\mathrm{sech}^2(x)$, the adjoint eigenfunction of f_0, (which is proportional to the integral (4.12) of $f_0(-x, \mu_2)$ given by (4.16)) namely,

$$(F_j, g_0) = 0 \quad j = 1, 2, 3, \ldots, \tag{A 9}$$

where (F, g) denotes the inner product defined by (4.9), and F_j stands for the right-hand side members in (A 5)–(A 7). At order $O(\epsilon^{\frac{1}{2}})$ the solvability condition (A 9) is evidently satisfied, since by (4.12) the integrand is a total differential, and f_1 can be determined by standard methods (such as by variation of parameters) as

$$f_1(x) = \tfrac{1}{4}\sigma_1\,\mathrm{sech}^2(x)\,[1-x\tanh(x)]. \tag{A 10}$$

The complementary solution of (A 5), i.e. f_0, need not be included in the expression for f_1, since it can always be absorbed in the leading term. At order $O(\epsilon)$ the solvability condition determines σ_1 as

$$\sigma_1 = s\,\|f_0\|^2/(g_0, f_1) = \sqrt{s}\,8/\sqrt{15}. \tag{A 11}$$

Here $\|\cdot\|$ is the norm induced by the inner product, $\|\cdot\|^2 \equiv (\cdot,\cdot)$, and the factor $\sqrt{s}$ in (A 11) arises from (g_0, f_1) being proportional to σ_1. Thus we see that σ_1 is real for

$\mu > 4$, ($s = +1$), but is purely imaginary for $\mu < 4$, i.e. $s = -1$. We further note that f_1 does not satisfy the integral constraint (4.8), and this is reflected at the next order by the fact that for f_2 we can match the regularity condition at only one of the boundaries at $\pm\infty$, say at $x = -\infty$. This can be seen by integrating (A 6) once and taking the limit as $x \to +\infty$,

$$f_2(x) \underset{x\to+\infty}{\longrightarrow} -\tfrac{1}{4}\sigma_1 \int_{-\infty}^{+\infty} f_1(x)\,\mathrm{d}x = -\tfrac{1}{16}\sigma_1^2. \tag{A 12}$$

The solvability condition for the $O(\epsilon^{\frac{3}{2}})$ problem determines the value of σ_2 as

$$\sigma_2 = -\sigma_1 (g_0, f_2)/(g_0, f_1). \tag{A 13}$$

To evaluate the numerator we do not need the explicit form of f_2 if we make use of the adjoint relationships, and the result is

$$\sigma_2 = -s\tfrac{4}{15}. \tag{A 14}$$

The eigenvalue thus has in a neighbourhood of $\mu = 4$, $\mu = 4 + s\epsilon$, the inner solution (4.22). As shown below, this inner expansion of σ and f can be matched well with the outer expansion so that the compound eigenfunction satisfies the regularity condition at $x = \pm\infty$.

(*b*) *The outer problem for the case of $\mu = 4$*

Since the expansion of f cannot satisfy the boundary conditions (4.6) at all orders, we look for an 'outer' problem by defining the multiscale outer variables as

$$x^+ = x(\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}})), \quad \tilde{x} = x\epsilon, \quad w(x^+, \tilde{x}) = f(x), \tag{A 15}$$

so that
$$\mathrm{d}/\mathrm{d}x = \epsilon^{\frac{1}{2}}\,\partial/\partial x^+ + \epsilon\,\partial/\partial\tilde{x} + O(\epsilon^{\frac{3}{2}}). \tag{A 16}$$

We take
$$w(x^+, \tilde{x}) = \epsilon w_2(x^+, \tilde{x}) + \epsilon^{\frac{3}{2}} w_3(x^+, \tilde{x}) + \dots, \tag{A 17}$$

and by substituting these expressions in (4.19), we obtain for the first two terms:

$O(\epsilon^{\frac{3}{2}})$
$$-4\,\partial w_2/\partial x^+ = \sigma_1 w_2, \tag{A 18}$$

$O(\epsilon^2)$
$$-4\,\partial w_3/\partial x^+ = \sigma_1 w_3 + 4\,\partial w_2/\partial\tilde{x} + \sigma_2 w_2, \tag{A 19}$$

where we have neglected the term $\operatorname{sech}^2(\epsilon^{-\frac{1}{2}}x^+)$, since for x^+ fixed it is exponentially small as $\epsilon \to 0$. Eliminating the secularity in (A 19), the first term in the outer expansion can be taken to be

$$w_2(x^+, \tilde{x}) = C \exp(-\tfrac{1}{4}\sigma_1 x^+ - \tfrac{1}{4}\sigma_2 \tilde{x}), \tag{A 20}$$

which satisfies the boundary condition $w_2 \to 0$ for x^+ and $\tilde{x} \to -\infty$ for both cases of $s = \pm 1$ by virtue of the expression (A 11) for σ_1, (A 14) for σ_2 and the relation $\tilde{x} = \epsilon^{\frac{1}{2}} x^+$. The constant C is determined by matching with the inner solution. By observing (A 12) and (4.6), we obtain a uniformly valid expansion up to order $O(\epsilon)$ (and up to a multiplicative constant) in the form

$$f(x;\mu) = f_0(x) + \epsilon^{\frac{1}{2}} f_1(x) + \epsilon\{f_2(x) - \tfrac{1}{16}\sigma_1^2 H(x)[\exp(-\tfrac{1}{4}\sigma_1 \epsilon^{\frac{1}{2}} x - \tfrac{1}{4}\sigma_2 \epsilon x) - 1]\} + O(\epsilon^{\frac{3}{2}}), \tag{A 21}$$

where $H(x)$ is the Heaviside step function. The eigenvalue corresponding to this eigenfunction is given by (4.22). The rate of growth of the perturbation (4.2) in the form of $\eta = f(x,\mu)\exp(\sigma t)$ with f given by (A 21) for $\mu \approx 4$, is seen quite different on the two sides of $\mu = 4$, being of $O(\epsilon)$ for $\mu < 4$ and of $O(\epsilon^{\frac{1}{2}})$ for $\mu > 4$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(c) *Inner and outer expansions for* $\mu = 1, 9$ *and other cases*

For the other two cases (4.15) and (4.17), the inner solution starts to violate the integral condition (4.8) at order $O(1)$ and the boundary conditions (4.6) at $O(\epsilon)$. The appropriate solution of the adjoint problem by (4.12), is

$$g_0(x;\mu_m) = \int_{-\infty}^{x} f_0(-s;\mu_m)\,\mathrm{d}s, \quad m = 1, 3, \tag{A 22}$$

and so it is in fact a distribution, being bounded but different from zero as $x \to +\infty$. The analysis relative to these cases is completely analogous to the one for $\mu = 4$, and we only quote the result of the uniform expansion for $\mu = \mu_m + s\epsilon$, to order $O(\epsilon)$:

$$f(x;\mu) = f_0(x;\mu_m) + \epsilon\{f_1(x;\mu_m) + (\chi_m\,\sigma_1/\mu_m)\,H(x)\,[1 - \exp(-(\sigma_1/\mu_m)\,\epsilon x)]\} + O(\epsilon^2), \tag{A 23}$$

where $\chi_m = \int_{-\infty}^{\infty} f_0(x;\mu_m)\,\mathrm{d}x$. We note that this expression not only satisfies the boundary conditions (4.6) to order $O(\epsilon)$ but is also consistent with the integral constraint (4.8) to the same order. The explicit form of f_1 is not important at this order, only the limits for $x \to \pm\infty$ are, and from (4.19) it can be shown that, for $m = 1, 3$

$$f_1(x;\mu_m) \underset{x\to-\infty}{\longrightarrow} 0, \quad f_1(x;\mu_m) \underset{x\to+\infty}{\longrightarrow} -\sigma_1\,\chi_m/\mu_m. \tag{A 24}$$

We find for the first term in the eigenvalue expansion

$$\sigma_1(\mu_m) = -(2s/\chi_m^2)\,(f_0, f_0), \tag{A 25}$$

which yields (4.23).

We notice that in (A 23) it is necessary to have $\sigma_1 > 0$ to meet the boundary condition at $+\infty$ and therefore by (A 25) we get eigenvalues of the form (4.23) for $\mu < \mu_m$, i.e. $s = -1$, only. Of course, this conclusion applies only to the branches originating from $\sigma = 0$ at $\mu = 1$ or 9 and there may exist other branches of eigenvalues issued from the other fixed point $\mu = 4$, as our numerical results indeed show.

Concerning the neighbourhood of $\mu = 0$, we notice that equation (4.14) admits for the forcing (i) at $\mu = 0$ the solution

$$f_0(x;0) = \tanh(x)\,[5\tanh^2(x) - 3], \tag{A 26}$$

which, however, violates the regularity conditions (4.6), and therefore is not an eigenfunction corresponding to $\sigma = 0$, as already noted. Nevertheless, for μ in a neighbourhood of 0, it might be possible to use this solution as an 'inner' one and to satisfy condition (4.6) by matching it with an 'outer' solution, which in this case is required already at order $O(1)$. Such an analysis is not pursued here, but is supplemented by the numerical calculations given in §4*b* and Appendix B.

For the case (ii) forcing, we fix $\mu = 4$ and let α vary, and we have the stationary solutions with $\sigma = 0$ for $\alpha = \nu(\nu+1)$, $\nu = 2, 3, \ldots$. The first two are

$$\alpha_0 = 6, \quad f_0(x;\alpha_0) = \mathrm{sech}^2(x), \tag{A 27}$$

$$\alpha_1 = 12, \quad f_0(x;\alpha_1) = \mathrm{sech}^2(x)\tanh(x). \tag{A 28}$$

The above procedure of determining the eigenvalues and eigenfunctions now with a perturbation of the parameter $\alpha = \alpha_\nu + s\epsilon$ can be applied here in close analogy. The presence of the first order term $-s(\mathrm{d}/\mathrm{d}x)[\operatorname{sech}^2(x) f_0(x;\alpha_n)]$ in equation of $O(\epsilon)$ in analogy with the term $s(\mathrm{d}/\mathrm{d}x) f_0(x;\mu_m)$ in (A 6) merely changes the numerical expressions for σ_1 and σ_2 (and the role played by the signature s). Thus we obtain the result for σ as

$$\sigma(\alpha_0+s\epsilon) = s\epsilon\tfrac{16}{5}+O(\epsilon^2), \tag{A 29}$$

$$\sigma(\alpha_1+s\epsilon) = \epsilon^{\frac{1}{2}}\frac{16\mathrm{i}}{\sqrt{105}}\mathrm{s}^{-\frac{1}{2}}+\epsilon\frac{16}{105}\mathrm{s}+O(\epsilon^{\frac{3}{2}}), \tag{A 30}$$

and the corresponding asymptotic expansion for the eigenfunctions shows that for $\nu = 0$ (or more general for ν even) there is no solution for $s = -1$, i.e. for $\alpha < \alpha_0$. Since by (A 29), σ has its leading term real, the forced solitary wave (2.5) is unstable for $\alpha > \alpha_0$, but the analysis on linear theory fails to provide any information for $\alpha < \alpha_0$. The expression (A 30) for σ shows that the stationary state ζ_s is unstable for α on both sides of α_1, growing at a rate of order $O(\epsilon^{\frac{1}{2}})$ for $\alpha < \alpha_1$, and at a slower rate of $O(\epsilon)$ for $\alpha > \alpha_1$. In the latter case, perturbations evolve with a periodic oscillation for $\alpha > \alpha_1$ due to the $O(\epsilon^{\frac{1}{2}})$ imaginary part of σ, much in analogy with $\mu > 4$ and $\mu < 4$ of case (i).

Appendix B. Global spectral analysis

The (numerically assisted) study of the power series expression for $f(z)$ in (4.25) provides accurate approximations to eigenvalues and eigenfunctions of the operator $\mathscr{L}_{\alpha,\mu}$, when α and μ are far from the special parameter values corresponding to $\sigma = 0$. In this Appendix we report the details of this analysis, where again, for definiteness, we choose to work with the forcing case (i), i.e. $\alpha = 12$.

The starting point is the expression (4.28) for $f(z)$. The algebra is simplified if we remove the singular behaviour at $z = 0$, 1 by setting

$$f(z) = [z^{\kappa_1}/(1-z)^{\kappa_1}]\,p(z), \tag{B 1}$$

so that (4.25) reduces to

$$\frac{\mathrm{d}^3p}{\mathrm{d}z^3}+\frac{Az+B}{z(1-z)}\frac{\mathrm{d}^2p}{\mathrm{d}z^2}+\frac{Cz^2+Dz+E}{z^2(1-z)^2}\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{Fz+G}{z^2(1-z)^2}p = 0, \tag{B 2}$$

where
$$A = -6, \quad B = 3(\kappa_1+1),$$
$$\left.\begin{aligned} &C = -6, \quad D = 6(1-\kappa_1), \quad E = 3\kappa_1(\kappa_1+1)+1-\tfrac{1}{4}\mu, \\ &F = -24, \quad G = 12(\kappa_1+1). \end{aligned}\right\} \tag{B 3}$$

The power series

$$p(z) = \sum_{n=0}^{\infty} a_n z^n \tag{B 4}$$

is a solution of (B 2) if the coefficients $\{a_n\}$ satisfy the second-order difference equation

$$Q(n)\,a_{n+1}+P(n)\,a_n+R(n)\,a_{n-1} = 0, \tag{B 5}$$

where

$$\left.\begin{aligned} Q(n) &\equiv (n+1)\,[(n-1)\,n+3(\kappa_1+1)\,n+3\kappa_1\,(k_1+1)+1-\tfrac{1}{4}\mu], \\ P(n) &\equiv n[-2(n-1)\,(n-2)-3(3+\kappa_1)\,(n-1)+6(1-\kappa_1)]+12(\kappa_1+1), \\ R(n) &\equiv (n-1)\,(n-2)\,(n-3)+6(n-1)\,(n-2)-6(n-1)-24. \end{aligned}\right\} \tag{B 6}$$

Let us first check if certain values of the parameters μ, σ exist such that the series terminates. For this to be the case, we must have

$$a_{N-1} \neq 0, \quad a_N = a_{N+1} = 0, \tag{B 7}$$

for some integer $N > 0$, by which (B 5) is reduced to

$$R(N)\, a_{N-1} = 0, \tag{B 8}$$

and since $R(n) = (n+1)(n+3)(n-4)$, this has only one solution,

$$N = 4. \tag{B 9}$$

Therefore $a_4 = 0$ and so

$$0 = \frac{P(3)}{Q(3)}a_3 + \frac{R(3)}{Q(3)}a_2 = \left[-a_2\frac{P(2)}{Q(2)} - a_1\frac{R(2)}{Q(2)}\right]\frac{P(3)}{Q(3)} + a_2\frac{R(3)}{Q(3)}$$

$$= \left[-a_1\frac{P(1)}{Q(1)} - a_0\frac{R(1)}{Q(1)}\right]\left[\frac{R(3)}{Q(3)} - \frac{P(3)}{Q(3)}\frac{P(2)}{Q(2)}\right] - a_1\frac{R(2)}{Q(2)}\frac{P(3)}{Q(3)}. \tag{B 10}$$

Since

$$a_1 = -[P(0)/Q(0)]\,a_0 \tag{B 11}$$

and $a_0 \neq 0$, the final equation for μ, σ becomes

$$\left[\frac{P(0)}{Q(0)}\frac{P(1)}{Q(1)} - \frac{R(1)}{Q(1)}\right]\left[\frac{R(2)}{Q(3)} - \frac{P(3)}{Q(3)}\frac{P(2)}{Q(2)}\right] + \frac{P(3)}{Q(3)}\frac{R(2)}{Q(2)}\frac{P(0)}{Q(0)} = 0. \tag{B 12}$$

After some lengthy calculations, this reduces to

$$(4\kappa_1^2 - \mu)(\mu - 4) = 0, \tag{B 13}$$

hence $\mu = 4$, any κ_1; or $\kappa_1 = \frac{1}{2}\sqrt{\mu}$ any $\mu > 0$. (B 14)

In the first case, solving for a_n, $n = 1, 2, 3$ in (B 5), we retrieve the solution found by Jeffrey & Kakutani (1972), i.e. a third-order polynomial in z times the factor $z^{\kappa_1}(1-z)^{-\kappa_1}$, which becomes, in the original independent variable x,

$$f(x) = \mathrm{e}^{2\kappa_j x}[\kappa_j(\kappa_j - 1)^2 + \mathrm{e}^{-x}\{\kappa_j(\kappa_j - 1) - (2\kappa_j - 1)\tanh x + \tanh^2 x\}], \tag{B 15}$$

for $j = 1, 2, 3$. The other case in (B 14) has already been considered in §4*a* for the perturbation expansions, since by (4.27) it implies $\sigma = 0$.

With the exception of the above two particular cases, the difference equation (B 5) does not seem to have a closed form solution in general, but it does show that the series (B 4) has a radius of convergence equal to 1. Therefore the eigenvalue σ can be calculated to any desired order of accuracy, limited only by the round-off error, by summing the two series in (4.28) numerically, one about $z = 0$ and the other about $z = 1$, and matching the two expansions at some intermediate point. By requiring continuity of the functions and the derivatives up to the second order, which of course can be determined analytically by differentiating the power series, the eigenvalues σ are therefore the roots of

$$\det\begin{vmatrix} p(z) & [(1-z)^{\kappa_1-\kappa_2}q_2(1-z)] & [(1-z)^{\kappa_1-\kappa_3}q_3(1-z)] \\ p'(z) & [(1-z)^{\kappa_1-\kappa_2}q_2(1-z)]' & [(1-z)^{\kappa_1-\kappa_3}q_3(1-z)]' \\ p''(z) & [(1-z)^{\kappa_1-\kappa_2}q_2(1-z)]'' & [(1-z)^{\kappa_1-\kappa_3}q_3(1-z)]'' \end{vmatrix} = 0, \tag{B 16}$$

where $(\cdot)'$ denotes derivative with respect to z, and the determinant depends on σ

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Table 1

s	ϵ	σ (by series)	σ (by perturbation theory)
		forcing (i) $\mu = 4+s\epsilon$	
-1	0.1	$0.10953\mathrm{i}+4.164\times10^{-3}$	$0.1088\mathrm{i}+4.\overline{44}\times10^{-3}$
-1	0.01	$0.03444\mathrm{i}+4.41\times10^{-4}$	$0.034426\mathrm{i}+4.\overline{44}\times10^{-4}$
$+1$	0.01	0.033967	0.033982
		forcing (i) $\mu = 9-\epsilon$	
	0.1	0.01426	0.01441
		forcing (ii) $\alpha = 12+s\epsilon$	
$+1$	0.1	$0.083184\mathrm{i}+2.4861\times10^{-3}$	$0.0823\mathrm{i}+2.54\times10^{-3}$
$+1$	0.01	$0.026052\mathrm{i}+2.534\times10^{-4}$	$0.026024\mathrm{i}+2.54\times10^{-4}$

through the indices κ_j. Here q_2 and q_3 are the power series obtained from h_2, and h_3 in (4.28) via multiplication by the (analytic at $z = 1$) factor $z^{-\kappa_1}$. Their coefficients can be easily determined by the difference equation (B 5), using $-\kappa_1$ rather than κ_1 in the definitions (B 3) and setting $n \rightarrow n+\kappa_1-\kappa_j$, $j = 2, 3$. Choosing a point z sufficiently far removed from 0 and 1, the power series involved in (B 16) have a fast convergence rate and the roots of equation (B 16) can be found by a Newton–Raphson scheme.

Table 1 provides a comparison between some of the eigenvalues found by the present numerical procedure and the corresponding ones evaluated by the perturbation techniques of the Appendix A.

Thus we see that in the domains of parameter values where both numerical and perturbation approach can be expected to be valid (i.e. ϵ not 'too small' for the power series approach and ϵ not 'too big' for the perturbation analysis) the agreement is quite satisfactory.

Appendix C. A hamiltonian system and its conservation laws

We briefly report here, for the convenience of the reader, some of the results mentioned in the text regarding the application of Noether's first theorem (Bogoliubov & Shirkov 1980, §2) to the evolution equations of interest.

Introducing the potential ϕ

$$\zeta(x,t) \equiv \phi_x(x,t), \tag{C 1}$$

a lagrangian for the fKdV equation (2.1) can be written as (see Whitham 1974, §16.14)

$$\mathscr{L}(\phi) = \frac{1}{2}\int_{-\infty}^{+\infty} [\phi_t\,\phi_x + \mu\phi_x^2 - 3\phi_x^3 + \phi_{xx}^2 - 6P(x)\,\phi_x]\,\mathrm{d}x, \tag{C 2}$$

as one can check by the Euler–Lagrange equations,

$$(\mathrm{d}/\mathrm{d}t)\,\delta\mathscr{L}/\delta\phi_t - \delta\mathscr{L}/\delta\phi = 0, \tag{C 3}$$

which reproduces equation (2.1), after using the similarity transformation (3.3). We note that (C 2) has a structure similar to a lagrangian describing the scattering of a field by an external one, whose evolution is not affected by the interaction (see Bogoliubov & Shirkov 1980, §24). By Noether's first theorem, if

$$\phi'(x',t') = \phi(x,t) + \delta\phi(x,t) \tag{C 4}$$

is a one-parameter transformation, ξ say, of the field ϕ and the coordinates x, t, for which,

$$\mathscr{L}'(\phi') = \mathscr{L}(\phi) + (\mathrm{d}/\mathrm{d}t)\,\mathscr{F}(\phi;\xi), \tag{C 5}$$

then the quantity

$$Q \equiv \left(\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\delta\mathscr{L}}{\delta\phi_t}\frac{\partial\phi'}{\partial\xi}\bigg|_{\xi=0}\right) - \frac{\partial\mathscr{F}}{\partial\xi}\bigg|_{\xi=0} \tag{C 6}$$

is a constant of motion. Now, from the structure of the lagrangian (C 2), it is obvious that we have invariance under the transformation

$$\phi'(x',t') = \phi(x,t)+\xi, \quad \mathscr{F}=0, \tag{C 7}$$

and so, according to (C 6),

$$\mathscr{M} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} \phi_x\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty} \zeta\,\mathrm{d}x \tag{C 8}$$

is conserved. The spatial translation

$$\phi'(x',t') = \phi(x+\xi,t) \tag{C 9}$$

does not yield the form (C 5) for the transformed lagrangian unless $P(x) =$ const., in which case it is easy to verify that $\mathscr{F}=0$ and the associated conserved quantity is

$$\mathscr{E} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} \phi_x^2\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty} \zeta^2\,\mathrm{d}x. \tag{C 10}$$

If the forcing P is independent of time, then translation with respect to time

$$\phi'(x',t') = \phi(x,t+\xi) \tag{C 11}$$

when ξ is infinitesimal, leads to a transformed lagrangian

$$\mathscr{L}'(\phi') = \mathscr{L}(\phi)+\xi(\mathrm{d}/\mathrm{d}t)\,\mathscr{L}(\phi)+O(\xi^2), \tag{C 12}$$

and so (C 5) holds with $\mathscr{F}\equiv\mathscr{L}$, and the associated conserved quantity is

$$\begin{aligned}\mathscr{H} &= \left(\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\delta\mathscr{L}}{\delta\phi_t}\frac{\partial\phi}{\partial t}\right) - \mathscr{L} = \frac{1}{2}\int_{-\infty}^{+\infty} [\mu\phi_x^2 - 3\phi_x^3 + \phi_{xx}^2 - 6P(x)\,\phi_x]\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} [\mu\zeta^2 - 3\zeta^3 + \zeta_x^2 - 6P(x)\,\zeta]\,\mathrm{d}x,\end{aligned} \tag{C 13}$$

which is the hamiltonian considered in the text. There are no other obvious invariances possessed by the lagrangian for a general forcing $P(x)$. For the regularized fKdV equation (5.27), the lagrangian (C 2) has to be modified into

$$\mathscr{L}_r = \frac{1}{2}\int_{-\infty}^{+\infty} [\phi_t(\phi_x-\phi_{xxx}) + 6(F-1)\,\phi_x^2 - 3\phi_x^3 + F\phi_{xx}^2 - 6P(x)\,\phi_x]\,\mathrm{d}x, \tag{C 14}$$

and the invariants corresponding to the one-parameter transformations considered above are

$$\mathscr{E}_r \equiv \frac{1}{2}\int_{-\infty}^{+\infty} [\zeta^2+\zeta_x^2]\,\mathrm{d}x, \tag{C 15}$$

and
$$\mathscr{H}_r \equiv \frac{1}{2}\int_{-\infty}^{+\infty} [6(F-1)\,\zeta^2 - 9\zeta^3 + F\zeta_x^2 - 6P(x)\zeta]\,\mathrm{d}x, \tag{C 16}$$

for (C 9), when P is a constant, and for (C 11), respectively.

References

Akylas, T. R. 1984 On the excitation of long nonlinear water waves by a moving pressure distribution. *J. Fluid Mech.* **141**, 455–466.

Benjamin, T. B. 1972 The stability of solitary waves. *Proc. R. Soc. Lond.* A **328**, 153–183.

Benjamin, T. B., Bona, J. L. & Mahony, J. J. 1972 Model equation for long waves in nonlinear dispersive systems. *Phil. Trans. R. Soc. Lond.* A **272**, 47–78.

Bogoliubov, N. N. & Shirkov, D. V. 1980 *Introduction to the theory of quantized fields*. New York: Wiley.

Bona, J. L. 1975 On the stability of solitary waves. *Proc. R. Soc. Lond.* A **344**, 363–374.

Bona, J. L. & Smith, R. 1975 The initial value problem for the Korteweg-de Vries equation. *Phil. Trans. R. Soc. Lond.* A **278**, 556–601.

Camassa, R. 1990 Ph.D. thesis, California Institute of Technology, Pasadena, California.

Camassa, R. & Wu, T. Y. 1991 Stability of some stationary solution for the forced KdV equation. *Physica* D. (In the press.)

Cole, S. L. 1985 Transient waves produced by flow past a bump. *Wave Motion* **7**, 579–587.

Ertekin, R. C. 1984 Soliton generation by moving disturbances in shallow water: theory, computation and experiments. Ph.D. thesis, University of California, Berkeley, California.

Ertekin, R. C., Webster, W. C. & Wehausen, J. V. 1986 Waves caused by a moving disturbance in a shallow channel of finite width. *J. Fluid Mech.* **177**, 49–65.

Grimshaw, R. H. J. & Smyth, N. 1986 Resonant flow of a stratified fluid over a topography. *J. Fluid. Mech.* **169**, 429–464.

Holm, D. D., Marsden, J. E., Ratiu, T. & Weinstein, A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123**, 1–116.

Jeffrey, A. & Kakutani, T. 1972 Weak nonlinear dispersive waves: a discussion centered around the Korteweg-de Vries equation. *SIAM Rev.* **14**, 582–643.

Kevorkian, J. & Yu, J. 1989 Passage through the critical Froude number for shallow-water waves over a variable bottom. *J. Fluid Mech.* **204**, 31–56.

Landau, L. & Lifshitz, E. 1958 *Quantum mechanics, nonrelativistic theory*. London: Pergamon.

Lee S. J. 1985 Generation of long waterwaves by moving disturbances. Ph.D. thesis, California Institute of Technology, Pasadena, California.

Lee, S. J., Yates, G. T. & Wu, T. Y. 1988 A theoretical and experimental study of precursor solitary waves generated by moving disturbances. In *Nonlinear water waves* (ed. K. Horikawa & H. Maruo), pp. 365–372. Springer-Verlag.

Lee, S. J., Yates, G. T. & Wu, T. Y. 1989 Experiments and analyses of upstream-advancing solitary waves generated by moving disturbances. *J. Fluid Mech.* **199**, 569–593.

Lighthill, M. J. 1967 Some special cases treated by the Whitham theory. *Proc. R. Soc. Lond.* A **299**, 28–53.

Malomed, B. A. 1988 Interaction of a moving dipole with a soliton in the KdV equation. *Physica* D **32**, 393–408.

Newell, A. C. 1985 *Solitons in mathematics and physics*. Philadelphia: SIAM.

Patoine, A. & Warn, T. 1982 The interaction of long, quasi-stationary baroclinic waves with topography. *J. atmos. Sci.* **39**, 1019–1025.

Teng, M. H. 1990 Forced emissions of nonlinear water waves in channels of arbitrary shape. Ph.D. thesis, California Institute of Technology, Pasadena, California.

Whitham, G. B. 1974 *Linear and nonlinear waves*. New York: Wiley.

Whitham, G. B. 1967 Variational methods and applications to water waves. *Proc. R. Soc. Lond.* A **299**, 6–25.

Wu, D. M. & Wu, T. Y. 1982 Three dimensional nonlinear long waves due to moving surface pressure. In *Proc. 14th Symp. on Naval Hydrodynamics, Washington, D.C.*, pp. 103–125.

Wu, D. M. & Wu, T. Y. 1988 Precursor solitons generated by three-dimensional disturbances moving in a channel. In *Nonlinear water waves* (ed. K. Horikawa & H. Maruo), pp. 69–75. Springer-Verlag.

Wu, T. Y. 1979 On tsunamis propagation-evaluation of existing models. In *Tsunamis Proc. National Science Foundation Workshop* (ed. L. S. Hwang & Y. K. Lee), pp. 110–149. Pasadena: Tetra Tech.

Wu, T. Y. 1981 Long waves in ocean and coastal waters. *J. Engng Mech. Div., Proc. ASCE* **107**, 502–522.

Wu, T. Y. 1985 On the generation of solitons. In *Symposium on Fluid Mechanics honoring Professor C.-S. Yih*. University of Michigan, Ann Arbor. 22–24 July 1985.

Wu, T. Y. 1987 On generation of solitary waves by moving disturbances. *J. Fluid Mech.* **184**, 75–99.

Wu, T. Y. 1988 Forced generation of solitary waves. In *Applied Mathematics, Astrophysics, A Symposium to Honor C. C. Lin, 22–24 June 1987*, MIT (ed. D. J. Benney, F. H. Shu & C. Yuan), pp. 198–212. Singapore: World Scientific.

Zabusky, N. J. & Kruskal, M. D. 1965 Interaction of solitons in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.* **15**, 241–243.

Received 2 October 1990; revised 19 April 1991; accepted 31 May 1991.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

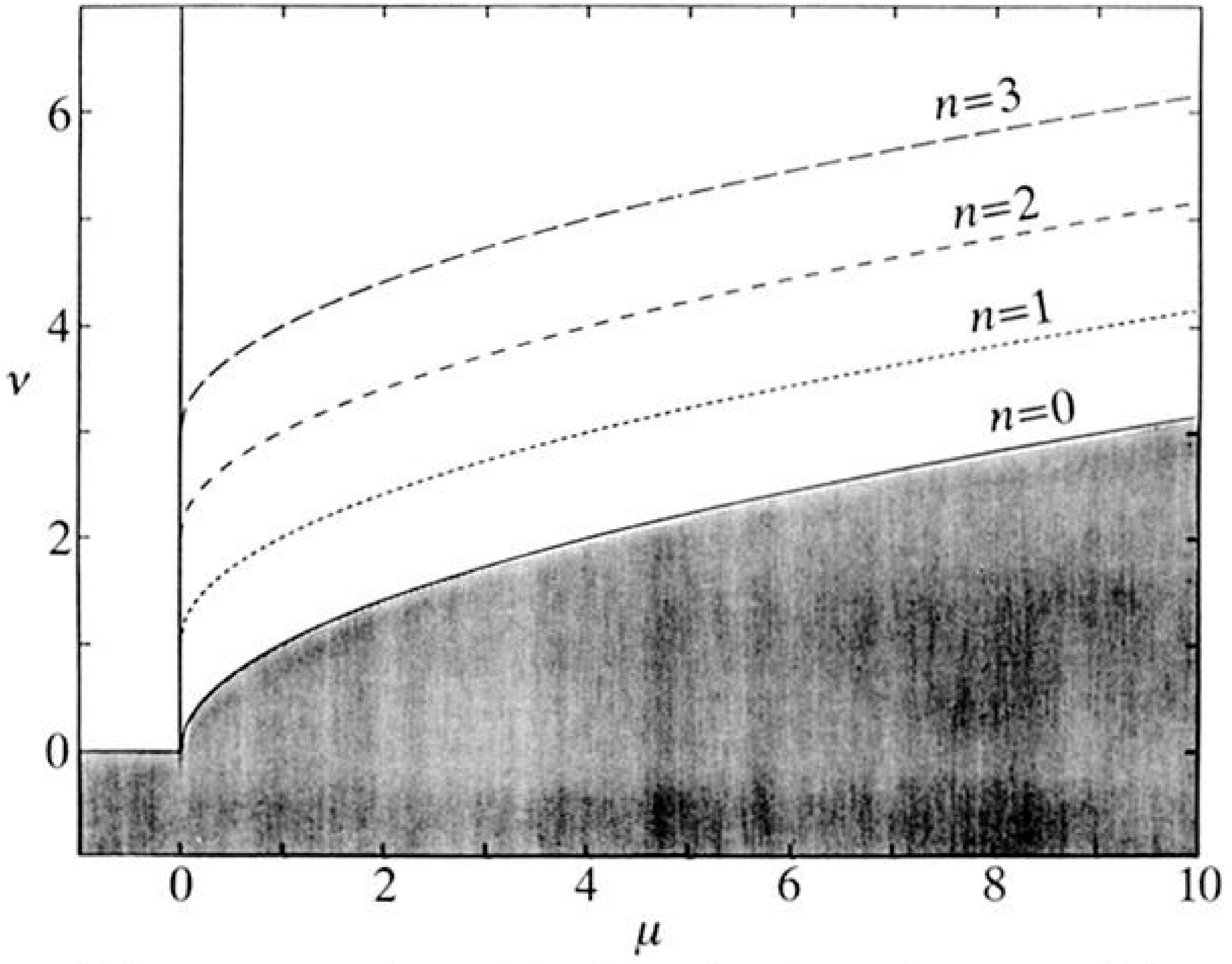


Figure 4. Subdivision of the parametric $\mu\nu(\alpha)$-plane for the existence of eigenvalues of the operator $\mathscr{L}_{\mu,\alpha}$. In the shaded region no eigenvalue with real part different from zero can exist.